AN

INTRODUCTION

TO

ALGEBRA;

WITH

NOTES AND OBSERVATIONS:

DESIGNED FOR THE USE OF

SCHOOLS, and Places of PUBLIC EDUCATION.

By JOHN BONNYCASTLE,

Author of the SCHOLAR'S GUIDE TO ARITHMETIC, and a TREATISE OF MENSURATION.

——Ingenuas didicisse fideliter artes
Emollit mores, nec sinit esse feros. OVID.

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E.P.

TO THE REV.

J. PRIESTLEY, L.L.D. F.R.S. &c.

SIR,

THE publication of the following treatiſe is owing to your kind encouragement and approbation; and I am happy to embrace this opportunity of teſtifying the high ſenſe I entertain of your condeſcending politeneſs and attention. Whilſt you are enlarging the bounds of ſcience by your numerous and important diſcoveries, you are equally ſolicitous of promoting every other laudable purſuit and uſeful undertaking. And to this amiable diſpoſition the world is no leſs indebted than to

your diſtinguiſhed and eminent abilities: the one commands our eſteem and regard, and the other our admiration. Permit me, therefore, Sir, as a ſincere tribute to your merit, to inſcribe to you this compendium, and to aſſure you that

I am,

With the higheſt reſpect,

Your moſt obedient and

obliged humble ſervant,

London,
Sep. 2, 1782.

JOHN BONNYCASTLE.

PREFACE.

THE powers of the mind, like thoſe of the body, are increaſed by continual exertion; application and induſtry ſupply the place of genius and invention; and even the creative faculty itſelf, may be ſtrengthened and improved by uſe and perſeverance. Uncultivated nature is uniformly rude and imbecile; and it is by imitation alone that we at firſt acquire knowledge, and the means of extending its bounds. A juſt and perfect acquaintance with the ſimple elements of ſcience, is a neceſſary ſtep towards our future progreſs and advancement; and this, aſſiſted by laborious inveſtigation and habitual inquiry, will conſtantly lead to eminence and perfection.

Books of rudiments, therefore, conciſely written, well digeſted, and methodically arranged, are treaſures of ineſtimable value; and too many attempts cannot be made to render them perfect and complete. When the firſt principles of any art or ſcience are firmly fixed and rooted in the mind, their application ſoon becomes eaſy,

pleaſant, and obvious; the underſtanding is delighted and enlarged; we conceive clearly, reaſon diſtinctly, and form juſt and ſatisfactory concluſions. But, on the contrary, when the mind, inſtead of repoſing on the ſtability of truth, and received principles, is wandering in doubt and uncertainty, our ideas will neceſſarily be confuſed and obſcure; and every ſtep we take, muſt be attended with freſh difficulties and endleſs perplexity.

That the grounds, or fundamental parts, of every ſcience, are dull and unentertaining, is a complaint univerſally made, and a truth not to be denied; but, then, what is obtained with difficulty is remembered with eaſe; and what is purchaſed with pain is poſſeſſed with pleaſure. The ſeeds of knowledge are ſown in every ſoil, but it is by proper culture alone that they are cheriſhed and brought to maturity. A few years of early and aſſiduous application never fails of procuring us the reward of our induſtry; and who is there, that knows the pleaſures and advantages which the ſciences afford, that would think his time miſ-ſpent, or his labours uſeleſs? Riches and honours are the gifts of fortune, caſually beſtowed or hereditarily received, and are frequently

abused by their possessors; but the superiority of wisdom and knowledge is a pre-eminence of merit, that originates with the man, and is the noblest of all distinctions.

Nature, bountiful and wise in all things, has provided us with an infinite variety of scenes, both for our instruction and entertainment; and, like a kind and indulgent parent, admits all her children to an equal participation of her blessings. But as the modes, situations, and circumstances of life are various, so accident, habit, and education, have each their predominating influence, and give to every mind its particular bias. Where examples of excellence are wanting, the attempts to attain it are but few; but eminence excites attention, and produces imitation. To raise the curiosity, and to awaken the listless and dormant powers of younger minds, we have only to point out to them a valuable acquisition, and the means of obtaining it. The active principles are immediately put into motion, and the certainty of the conquest is insured from a determination to conquer.

But of all the sciences which serve to call forth this spirit of enterprise and inquiry, there are none more eminently useful than the Mathematics.

By an early attachment to theſe elegant and ſublime ſtudies, we acquire a habit of reaſoning, and an elevation of thought, that fixes the mind, and prepares it for every other purſuit. From a few ſimple axioms, and evident principles, we proceed gradually to the moſt general propoſitions, and remote analogies; deducing one truth from another, in a chain of argument, well connected and logically purſued; that brings us at laſt, in the moſt ſatisfactory manner, to the concluſion, and ſerves as a general direction in all our inquiries after truth.

And it is not only in this reſpect that mathematical learning is ſo highly valuable: it is, likewiſe, equally eſtimable for its practical utility. Almoſt all the works of art, and devices of man, have a dependence upon its principles, and are indebted to it for their origin and perfection. The cultivation of theſe admirable ſciences is, therefore, a thing of the utmoſt importance, and ought to be conſidered as a principal part of every liberal and well regulated plan of education. They are the guide of our youth, the perfection of our reaſon, and the foundation, or baſis, of every great and noble undertaking.

From theſe conſiderations, I have been induced to undertake an introductory courſe of mathematical ſcience; and, from the kind encouragement I have hitherto received, am not without hopes of a continuance of the ſame candour and approbation. Conſiderable practice as a teacher, and a long attention to the difficulties and obſtructions which retard the progreſs of learners in general, has enabled me to accommodate myſelf the more eaſily to their capacities and underſtandings. And as an earneſt deſire of promoting and diffuſing uſeful knowledge, is the chief motive for this undertaking, ſo no pains, or attention, ſhall be wanting to make it as complete and perfect as poſſible.

The ſubject of the preſent performance is ALGEBRA; which is one of the moſt important and uſeful branches of thoſe ſciences; and may be juſtly conſidered as the key to all the reſt. Geometry delights us by the ſimplicity of its principles, and the elegance of its demonſtrations: Arithmetic is confined in its object, and partial in its application: but Algebra, or the Analytic Art, is general and comprehenſive, and may be uſed with ſucceſs, in all caſes, where truth is to be obtained, and proper data can be eſtabliſhed.

To trace this noble ſcience to its birth, and to point out all the various alterations and improvements it has undergone in its progreſs, would far exceed the limits of a preface. It will be ſufficient to obſerve, that the invention is of the higheſt antiquity, and has challenged the praiſe and admiration of all ages. *Diophantus* appears to have been the firſt, among the ancients, who applied it to the ſolution of indeterminate and unlimited problems; but it is to the moderns that we are principally indebted for all the moſt curious refinements of the art, and its great and extenſive uſefulneſs in every abſtruſe and difficult inquiry. *Newton*, *Maclaurin*, *Saunderſon*, *Simpſon*, and *Emerſon*, are thoſe, of our own countrymen, who have particularly excelled in this reſpect; and it is to their works that I would refer the young ſtudent, as the patterns of elegance and perfection.

The following compendium is formed entirely upon the model of thoſe writers, and is intended as a uſeful and neceſſary introduction to them. Almoſt every ſubject, that belongs to pure Algebra, is conciſely and diſtinctly treated of; and no pains have been ſpared, to make the whole as eaſy and intelligible as poſſible. A great number of elementary books have already been written

+ The reader may here see how far ev[...]
own productions. What a Construct[...]
of our best modern Authors (by say[...]
elegant manner, nay farther, a ch[...]
stand no ways consonant to the Rem[...]

upon this subject; but there are none, that I have yet seen, but what appear to me to be extremely defective. Besides being totally unfit for the purpose of teaching, they are generally calculated to vitiate the taste, and mislead the judgment. A tedious and inelegant method prevails through the whole, so that the beauty of the science is destroyed by the clumsy and aukward manner in which it is treated; and the learner, when he is afterwards introduced to some of our best writers, is obliged to unlearn and forget every thing that he has been at so much pains in acquiring. ‡

It is in the sciences as in every branch of polite literature; there is a certain taste and elegance that is only to be obtained from the best authors, and a judicious use of their instructions. To direct the student in his choice of books, and to prepare him properly for the advantages he may receive from them, is, therefore, the business of every writer who engages in the humble, but useful, task of a preliminary tutor. This information I have been careful to give, in every part of the present performance, where it could be thought to be in the least useful, or necessary. The nature and confined limits of my plan,

...thor pushes his Arguments to enforce his
...ey Mr. Bonnycastle here put on the Writings
...ey handle their subjects in a tedious & in-
...e aukward manner! This is mechanical
...a Simpson, an Emerson, an Newton. He as

any one will testify (I have no doubt) who hath perused
read their Writings,

admitted not of diffuſe obſervations, or a formal enumeration of particalars; but nothing of real uſe and importance has been omitted. My principal object was to conſult the eaſe, ſatisfaction, and accommodation of the learner, and if the execution of the work is found equal to the deſign, my purpoſe will be anſwered, and it cannot fail of meeting with a candid and favourable reception from the public.

ERRATA.

Page 12, line 7, for xy^4 read x^4-y^4. P. 20, l. 13, for $2bx^2$ read $2bx$. P. 22, l. 3, for $a+b$ read $x+b$. P. 27, l. 13, for $\frac{5bx}{a}$ read $\frac{5bx}{2a}$. P. 29, l. 29, dele —. P. 31, l. 20, for $x+b$ read $x+a$. P. 32, l. 10, dele —. P. 49, l. 5, for x^5 read x^6. P. 50, l. 24 and 26, read $a\times\sqrt[3]{a}$, $3a^3\times\sqrt[3]{a}$, $9a^5\times\sqrt[3]{a}$, &c. P. 54, l. 33, for $+$ read $::$. P. 56, l. 11, for 19 read 16. P. 60, l. 24, for $+3x$ read $-3x$, and l. ult. for ba read $6a$. P. 64, l. 9, for 2 read 3. P. 79, l. 26, dele $\surd$. P. 82, l. 21, for $x-$ read $x=-$, and line ult. for negative read affirmative.

ALGEBRA.

DEFINITIONS.

ALGEBRA is the art of computing by symbols.

1. *Like quantities* are those that consist of the same letters.

2. *Unlike quantities* are those that consist of different letters.

3. *Given quantities* are those whose values are known.

4. *Unknown quantities* are those whose values are unknown.

5. *Simple quantities* are those that consist of one term only.

6. *Compound quantities* are those that consist of several terms.

7. *Positive* or *affirmative quantities* are those to be added.

8. *Negative quantities* are those to be subtracted.

9. *Like signs* are all + or all −.

10. *Unlike signs* are + and −.

11. *The co-efficient* of any quantity is the number prefixed to it.

12. *A binomial quantity* is one consisting of two terms; a *trinomial* of three terms; and a *quadrinomial* of four terms; &c.

13. *A residual quantity* is a binomial where one of the terms is negative.

14. *The power of a quantity* is its square, cube, biquadrate &c.

15. *The index* or *exponent* is the number expressing the power to which the quantity is involved.

16. *A rational quantity* is that which has no radical sign.

17. *The reciprocal* of any quantity is that quantity inverted, or unity divided thereby.

Explanation of the Characters.

$+$	Is the sign	of addition.
$-$	——	of subtraction.
$\times$	——	of multiplication.
$\div$	——	of division.
$\sqrt{}$	——	of the square root.
$\sqrt[3]{}$	——	of the cube root.
$\sqrt[m]{}$	——	of the *m* root.
$=$	——	of equality.

Thus $a+b$ is the sum of a and b.

$a-b$ is the difference of a and b.

$a \backsim b$ is the difference of a and b when it is not known which is the greatest.

ab, or $a \times b$, or $a.b$ is the product of a and b.

$a \div b$ or $\frac{a}{b}$ is a divided by b.

$\sqrt{a}$ or $a^{\frac{1}{2}}$ is the square root of a.

$\sqrt[3]{a}$ or $a^{\frac{1}{3}}$ is the cube root of a.

a^2 is the square of a.

a^3 is the cube of a.

a^m is the m power of a.

$a^{\frac{1}{m}}$ is the m root of a.

$\frac{1}{a}$ is the reciprocal of a, and $\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$.

In the computation of problems, the first letters of the alphabet are put for known quantities, and the last letters for those that are unknown.

AXIOMS.

1. If equal quantities be added to equal quantities the wholes will be equal.
2. If equal quantities be taken from equal quantities, the remainders will be equal.
3. If equal quantities be multiplied by equal quantities, the products will be equal.
4. If equal quantities be divided by equal quantities, the quotients will be equal.
5. The equal powers or roots of equal quantities are equal.
6. Two quantities respectively equal to a third, are equal to each other.
7. The whole is equal to all its parts taken together.

ADDITION.

CASE I.

To add quantities that are like, and have like ſigns.

RULE.

Add all the co-efficients together, and to their ſum adjoin the letters common to each term, prefixing the common ſign.

EXAMPLES*:

$$
\begin{array}{r}
5a\\
7a\\
8a\\
10a\\
2a\\
a\\
\hline
33a\\
\hline
\end{array}
\qquad
\begin{array}{r}
-6bx\\
-3bx\\
-2bx\\
-7bx\\
-bx\\
-5bx\\
\hline
-24bx\\
\hline
\end{array}
\qquad
\begin{array}{r}
8bxy\\
7bxy\\
3bxy\\
4bxy\\
5bxy\\
bxy\\
\hline
28bxy\\
\hline
\end{array}
$$

$$
\begin{array}{r}
5x^2+xy\\
3x^2+2xy\\
x^2+3xy\\
7x^2+8xy\\
x^2+xy\\
\hline
17x^2+15xy\\
\hline
\end{array}
\qquad
\begin{array}{r}
7ax-y\\
8ax-3y\\
6ax-2y\\
4ax-3y\\
ax-y\\
\hline
26ax-10y\\
\hline
\end{array}
$$

* When a leading quantity has no ſign before it + is always underſtood; and a quantity without any co-efficient prefixed to it is ſuppoſed to have 1, or unity.

$$\begin{array}{lll}
15xy & -8y^2 & 7a-6b \\
2xy & -7y^2 & 4a-3b \\
7xy & -\ \ y^2 & 2a-8b \\
xy & -6y^2 & a-\ \ b \\
xy & -\ \ y^2 & 3a-2b \\
\hline
\end{array}$$

$$\begin{array}{ll}
3x^{\frac{1}{2}}-\ \ xy & 3xy-\ \ x+2ab \\
2x^{\frac{1}{2}}-3xy & 2xy-3x+2ab \\
4x^{\frac{1}{2}}-8xy & 2xy-4x+8ab \\
x^{\frac{1}{2}}-2xy & 5xy-3x+\ \ ab \\
\hline
\end{array}$$

CASE II.

To add quantities that are like, but have unlike signs.

RULE.

1. Add all the affirmative co-efficients into one sum, and all the negative ones into another.
2. Subtract the least sum from the greatest, and to the difference prefix the sign of the greatest, with the common quantity.

EXAMPLES:

$$\begin{array}{r}
-3a \\
+7a \\
+8a \\
-a \\
-2a \\
\hline
+9a \\
\hline
\end{array}
\qquad
\begin{array}{r}
+8ax^2 \\
+7ax^2 \\
-3ax^2 \\
-4ax^2 \\
+4ax^2 \\
\hline
+12ax^2 \\
\hline
\end{array}
\qquad
\begin{array}{r}
+6x^{\frac{1}{3}}+8y \\
-3x^{\frac{1}{3}}+7y \\
-13x^{\frac{1}{3}}+8y \\
+2x^{\frac{1}{3}}-3y \\
+x^{\frac{1}{3}}-y \\
\hline
-7x^{\frac{1}{3}}+19y \\
\hline
\end{array}$$

$$\begin{array}{r}
-2xy+8 \\
-3xy+7 \\
+xy-10 \\
+5xy-7 \\
\hline
+xy-2 \\
\hline
\end{array}
\qquad
\begin{array}{r}
+8x^2-y+3\sqrt{x} \\
-10x^2-3y+2\sqrt{x} \\
-4x^2-2y+\sqrt{x} \\
+9x^2+6y-10\sqrt{x} \\
\hline
+3x^2 \cdots * -4\sqrt{x} \\
\hline
\end{array}$$

$$\begin{array}{r}
-2a^2 \\
-3a^2 \\
-8a^2 \\
+10a^2 \\
+13a^2 \\
\hline
\\
\hline
\end{array}
\qquad
\begin{array}{r}
+8b^2y^3 \\
+6b^2y^3 \\
-10b^2y^3 \\
-20b^2y^3 \\
-b^2y^3 \\
\hline
\\
\hline
\end{array}
\qquad
\begin{array}{r}
-3ab+3 \\
+8ab-10 \\
+3ab-6 \\
-ab+2 \\
-2ab+11 \\
\hline
\\
\hline
\end{array}$$

$$\begin{array}{r}
+4\sqrt{bx}-x \\
-3\sqrt{bx}-3x \\
-\sqrt{bx}-8x \\
-6\sqrt{bx}+11x \\
\hline
\\
\hline
\end{array}
\qquad
\begin{array}{r}
+4x^2-y^2-2 \\
+10x^2-8y^2-4 \\
-6x^2-y^2+8 \\
+3x^2+10y^2-1 \\
\hline
\\
\hline
\end{array}$$

CASE III.

To add quantities that are unlike, and have unlike signs.

RULE.

Collect the like quantities together by the last rule, and set down those that are unlike, one after another, with their proper signs.

EXAMPLES:

$$\begin{array}{r}
2x \\
3y \\
-a \\
x^2 \\
\hline
2x+3y-a+x^2 \\
\hline
\end{array}
\qquad
\begin{array}{l}
2x-x^2 \\
3a-x \\
2ax+6x^2 \\
3\sqrt{x}-2ax \\
\hline
x+5x^2+3a+3\sqrt{x} \\
\hline
\end{array}$$

$$\begin{array}{r}
12ax-x^2-6+\sqrt{ax-x^2} \\
-6ax+x^2-x+10 \\
3y-ax-4-2\sqrt{ax-x^2} \\
\hline
5ax-\sqrt{ax-x^2}-x+3y \\
\hline
\end{array}$$

$$\begin{array}{r}
3x^2y \\
-2xy \\
-3y^2x \\
-8x^2y \\
\hline
\\
\hline
\end{array}
\qquad
\begin{array}{l}
2\sqrt{x}-8 \\
3\sqrt{xy}+10 \\
6ab+x \\
-8+\sqrt{xy} \\
\hline
\\
\hline
\end{array}
\qquad
\begin{array}{r}
3a^2-8+x^{\frac{1}{2}}-2a \\
3a-10+a^2-6a \\
x^2-4a^2+8-4\sqrt{x} \\
10-a-x^2-x^{\frac{1}{3}} \\
\hline
\\
\hline
\end{array}$$

SUBTRACTION.

RULE.

Change the ſigns of all the quantities to be ſubtracted, and then add them together as in addition, and the reſult will be the remainder required.

EXAMPLES:

$$\begin{array}{r} 3a^2-2b \\ 2a^2-3b \\ \hline a^2+b \\ \hline \end{array} \qquad \begin{array}{r} 6x^2-8y+2 \\ x^2+9y-20 \\ \hline 5x^2-17y+22 \\ \hline \end{array} \qquad \begin{array}{r} 35xy-2+8x-y^{\frac{1}{2}} \\ 24xy-8-8x-3y \\ \hline 11xy+6+16x-y^{\frac{1}{2}}+3y \\ \hline \end{array}$$

$$\begin{array}{r} 8ax-2\sqrt{xy}-10 \\ 10x-6\sqrt{xy}-ax \\ \hline 9ax+4\sqrt{xy}-10-10x \\ \hline \end{array} \qquad \begin{array}{r} 4\sqrt{\frac{x}{a}}-10-8x-3xy \\ -5xy-7x+3-y \\ \hline 4\sqrt{\frac{x}{a}}-13-x+2xy+y \\ \hline \end{array}$$

$$\begin{array}{r} 5x^2y-8 \\ -3x^2y+10 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 4\sqrt{xy}-x\sqrt{xy} \\ 2\sqrt{xy}+12+xy \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 5x^2y^2+\sqrt{\frac{x}{a}}-8-4b \\ 6x^2y^2-10+4b-\sqrt{\frac{-x}{2}} \\ \hline \\ \hline \end{array}$$

MULTIPLICATION.

CASE I.

To multiply simple quantities.

RULE.

Multiply the co-efficients of the two terms together, and to the product prefix all the letters in those terms, and the result will be the whole product required.

Note. Like signs produce +, and unlike signs −.

EXAMPLES:

$2a$	$-2a$	$5a$	$-9x$
$3b$	$+4b$	$-6x$	$-5b$
$6ab$	$-8ab$	$-30ax$	$+45bx$

$7ab$	$6a^2x$	$-x^2y$	$-7xy$
$-5ac$	$5x$	xy^2	$-xy$
$-35a^2bc$	$30a^2x^2$	$-x^3y^3$	$+7x^2y^2$

a	$-ax$	$+5xy$	$-xy$
x	$-7b$	-3	$+6ax$

CASE II.

When one of the factors is a compound quantity.

RULE.

Find the products of the multiplier, and every particular term of the multiplicand separately, and place them one after another, with their proper signs, and the result will be the whole product required.

EXAMPLES:

$4a-2b$	$6xy-8$	$8a^2-2x+6$
$3a$	$2x$	$3xy$
$12a^2-6ab$	$12x^2y-16x$	$24a^2xy-6x^2y+18xy$

$3x-a$	$5x-7a$	$3y-8+2xy$
$2a$	$-x$	xy

CASE III.

When both the factors are compound quantities.

RULE.

Multiply every particular term of the multiplier into every term of the multiplicand respectively, and set down the products one after another with their proper signs, and their sum will be the whole product required.

EXAMPLES:

$$\begin{array}{l} x+y \\ x+y \\ \hline x^2+xy \\ \quad +xy+y^2 \\ \hline x^2+2xy+y^2 \\ \hline \end{array} \qquad \begin{array}{l} x+y \\ x-y \\ \hline x^2+xy \\ \quad -xy-y^2 \\ \hline x^2 \quad * \quad -y^2 \\ \hline \end{array} \qquad \begin{array}{l} x-y \\ x-y \\ \hline x^2-xy \\ \quad -xy+y^2 \\ \hline x^2-2xy+y^2 \\ \hline \end{array}$$

$$\begin{array}{l} 5x+4y \\ 3x-2y \\ \hline 15x^2+12xy \\ \quad -10xy-8y^2 \\ \hline 15x^2+2xy-8y^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+xy-y^2 \\ x-y \\ \hline x^3+x^2y-xy^2 \\ \quad -x^2y-xy^2+y^3 \\ \hline x^3 \quad * \quad 2xy^2+y^3 \\ \hline \end{array}$$

$$\begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3 \quad * \quad * \quad -y^3 \\ \hline \end{array} \qquad \begin{array}{l} x+y-z \\ x-y+z \\ \hline x^2+xy-xz \\ \quad -xy-y^2+yz \\ \quad\quad +xz+yz-z^2 \\ \hline x^2 \quad * \quad -y^2+2yz-z^2 \\ \hline \end{array}$$

$$\begin{array}{l} 3x^2-2xy+5 \\ x^2+2xy-3 \\ \hline 3x^4-2x^3y+5x^2 \\ \quad +6x^3y-4x^2y^2+10xy \\ \quad\quad -9x^2 \quad + \quad 6xy-15 \\ \hline 3x^4+4x^3y-4x^2-4x^2y^2+16xy-15 \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. *Product* $36a^2x$
2. Multiply $4x^2-2y$ by $2y$. *Product* $8x^2y-4y^2$
3. Multiply $2x+4y$ by $2x-4y$. *Product* $4x^2-16y^2$
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Product* $x\;y^4$
5. Multiply x^2+xy+y^2 by x^2-xy+y^2. *Product* $x^4+x^2y^2+y^4$
6. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$. *Product* $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$

DIVISION.

CASE I.

When the divisor is a simple quantity.

RULE.

1. Place the dividend above a small line, and the divisor under it, in the manner of a vulgar fraction.

2. Expunge those letters that are common to both the dividend and divisor, and divide the co-efficients of all the terms by any number that will divide them without a remainder, and the result will be the quotient required.

Note. Like signs make +, and unlike signs —, the same as in multiplication.

EXAMPLES:

$\frac{a}{a}=1$; $\frac{8bc}{2b}=4c$; $\frac{abc}{bcd}=\frac{a}{d}$; $\frac{10ab+15ac}{20ad}=\frac{2b+3c}{4d}$;

$\frac{ab+b^2}{2b}=\frac{a+b}{2}$; $\frac{12xy}{6x^2}=\frac{2y}{x}$; $\frac{30ax-54ay}{12ab}=\frac{5x-9y}{2b}$;

$\frac{10x^2y-15y^2-5y}{5y}=2x^2-3y-1$.

1. Divide $18x^2$ by $9x$. *Quotient* $2x$.
2. Divide $10x^2y^2$ by $-5x^2y$. *Quotient* $-2y$
3. Divide $-9ax^2y^2$ by $9x^2y$. *Quotient* $-ay$
4. Divide $-8x^2$ by $-2x$. *Quotient* $+4x$
5. Divide $a+3ax-x^2$ by a. *Quotient* $1+3x-\frac{x^2}{a}$
6. Divide $3a^2-15+6a+3b$ by $3a$.

Quotient $a-\frac{5}{a}+2+\frac{b}{a}$.

CASE II.

When the divisor and dividend are both compound quantities.

RULE.

1. Range the terms of both the quantities according to the dimensions of some letter in them; so that the first term may have the highest power of that letter, and the second term the next highest power; and so on.

2. Divide the first term of the dividend by the first term of the divisor, and place the result in the quotient.

3. Multiply the whole divisor by the quotient term last found, and subtract the result from the dividend.

4. To this remainder bring down the next term of the dividend, and divide as before; and so on, as in common arithmetic.

Note, Like signs produce $+$, and unlike signs $-$.

EXAMPLES:

$$
\begin{array}{l}
a+x)\,a^3+5a^2x+5ax^2+x^3\,(a^2+4ax+x^2 \\
\underline{a^3+a^2x} \\
4a^2x+5ax^2 \\
\underline{4a^2x+4ax^2} \\
ax^2+x^3 \\
\underline{ax^2+x^3} \\
*
\end{array}
$$

$$
\begin{array}{l}
x-3)\,x^3-9x^2+27x-27\,(x^2-6x+9 \\
\underline{x^3-3x^2} \\
-6x^2+27x \\
\underline{-6x^2+18x} \\
9x-27 \\
\underline{9x-27} \\
*
\end{array}
$$

$$
\begin{array}{l}
a-x)\,a^3-x^3\,(a^2+ax+x^2 \\
\quad a^3-a^2x \\
\hline
\qquad a^2x-x^3 \\
\qquad a^2x-ax^2 \\
\hline
\qquad\quad ax^2-x^3 \\
\qquad\quad ax^2-x^3 \\
\hline
\qquad\qquad *
\end{array}
$$

$$
\begin{array}{l}
b-y)\,b^4-y^4\,(b^3+b^2y+by^2+y^3 \\
\quad b^4-b^3y \\
\hline
\qquad b^3y-y^4 \\
\qquad b^3y-b^2y^2 \\
\hline
\qquad\quad b^2y^2-y^4 \\
\qquad\quad b^2y^2-by^3 \\
\hline
\qquad\qquad by^3-y^4 \\
\qquad\qquad by^3-y^4 \\
\hline
\qquad\qquad\quad *
\end{array}
$$

EXAMPLES FOR PRACTICE.

1. Divide $a^2+2ax+x^2$ by $a+x$. *Quotient* $a+x$
2. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$.
Quotient $a^2-2ay+y^2$
3. Divide 1 by $1-x$. *Quotient* $1+x+x^2+x^3$ &c.
4. Divide $6x^4-96$ by $3x-6$.
Quotient $2x^3+4x^2+8x+16$
5. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$. *Quotient* $a^3-3a^2x+3ax^2-x^3$

ALGEBRAIC FRACTIONS.

PROBLEM I.

To reduce a mixed quantity to an improper fraction.

RULE.

Multiply the integer by the denominator of the fraction, and to the product add the numerator; and the denominator being placed under this sum will give the improper fraction required.

EXAMPLES:

$$3\frac{5}{7}=\frac{3\times7+5}{7}=\frac{26}{7}; \qquad x+\frac{x^2}{a}=\frac{ax+x^2}{a};$$

$$a-\frac{b}{c}=\frac{ac-b}{c}; \qquad 5\frac{2}{3}=\frac{5\times3+2}{3}=\frac{17}{3};$$

$$1-\frac{2x}{a}=\frac{a-2x}{a}; \qquad a-x+\frac{a^2-ax}{x}=\frac{a^2-x^2}{x}.$$

Let the following mixed quantities be reduced to improper fractions:

$$8\frac{6}{7}; \quad a^2-\frac{2a}{b}; \quad 1+2x-\frac{x-3}{5x}; \quad b+\frac{1-x+a}{c};$$

PROBLEM II.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator for the integral part, and place the remainder over the denominator for the fractional part, and it will be the mixed quantity required.

EXAMPLES:

$$\frac{17}{5}=3\frac{2}{5};\ \frac{ax+a^2}{x}=a+\frac{a^2}{x};\ \frac{ay+2y^2}{a+y}=y+\frac{y^2}{a+y};$$

$$\frac{25}{4}=6\frac{1}{4};\ \frac{ab-a^2}{b}=a-\frac{a^2}{b};\ \frac{a^2+x^2}{a-x}=a+x+\frac{2x^2}{a-x}.$$

Let the following improper fractions be reduced to mixed quantities.

$$\frac{35}{8};\ \frac{28}{4};\ \frac{3ab-b^2}{a};\ \frac{2x^2y}{2x};\ \frac{b-2x^2}{b+x};$$

PROBLEM III.

To reduce fractions of different denominators, to those of the same value that shall have a common denominator.

RULE.

Multiply every numerator ſeparately into all the denominators but its own for the new numerators, and all the denominators together for the common denominator required.

EXAMPLES:

1. Reduce $\frac{a}{b}$ *and* $\frac{b}{c}$ to fractions of equal values that ſhall have a common denominator.

$$\begin{array}{l} a \times c = ac \\ b \times b = b^2 \\ \hline b \times c = bc \end{array}$$

$\frac{ac}{bc}$ *and* $\frac{b^2}{bc}$ = *fractions required.*

2. Reduce $\frac{a}{b}$, $\frac{b}{c}$ *and* $\frac{c}{d}$ to equivalent fractions, having a common denominator.

$$\begin{array}{l} a \times c \times d = acd \\ b \times b \times d = b^2d \\ c \times b \times c = c^2b \\ \hline b \times c \times d = bcd \end{array}$$

$\frac{acd}{bcd}$, $\frac{b^2d}{bcd}$, *and* $\frac{c^2b}{bcd}$ = *fractions required.*

3. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator. *Anſ.* $\frac{2xc}{ac}$ *and* $\frac{ab}{ac}$.

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to a common denominator. *Anſ.* $\frac{ac}{bc}$ *and* $\frac{ab+b^2}{bc}$.

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d to a common denominator. *Anſ.* $\frac{9xc}{6ac}$, $\frac{4ab}{6ac}$, *and* $\frac{6adc}{6ac}$.

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{2x}{a}$ to a common denominator. *Anſ.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$ *and* $\frac{12a^2+24x}{12a}$.

PROBLEM IV.

To find the greateſt common meaſure of a fraction.

RULE.

1. Range the quantities according to the dimenſions of ſome letter, as is ſhewn in diviſion.

2. Divide the greater term by the leſs, and the laſt diviſor by the laſt remainder, and ſo on till nothing remains; and the diviſor laſt uſed will be the common meaſure required.

Note, All the letters or figures that are common to each diviſor, muſt be thrown out of them before they are uſed in the operation.

EXAMPLES:

1. To find the greatest common measure of $\frac{cx+x^2}{ca^2+a^2x}$.

$$cx+x^2)ca^2+a^2x$$
$$\text{or } c+x\ )ca^2+a^2x(a^2$$
$$ca^2+a^2x$$
$$\overline{\quad * \quad}$$

Therefore the greatest common measure is $c+x$.

2. To find the greatest common measure of $\frac{x^3-b^2x}{x^2+2bx^2+b^2}$.

$$x^2+2bx+b^2)x^3-b^2x(x$$
$$x^3+2bx^2+b^2x$$
$$\overline{-2bx^2-2b^2x)x^2+2bx+b^2}$$
$$\text{or}\quad x+b\quad)x^2+2bx+b^2(x+b$$
$$x^2+bx$$
$$\overline{bx+b^2}$$
$$bx+b^2$$
$$\overline{\quad * \quad}$$

Therefore $x+b$ *is the greatest common divisor.*

3. To find the greatest common divisor of $\frac{x^4-b^4}{x^3-b^2x^3}$. *Ans.* x^2-b^2.

4. To find the greatest common measure of $\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$. *Ans.* $a+b$.

PROBLEM V.

To reduce a fraction to its lowest terms.

RULE.

1. Find the greatest common measure, as in the last problem.

2. Divide both the terms of the fraction by the common measure thus found, and it will reduce the fraction as required.

EXAMPLES:

1. Reduce $\frac{cx+x^2}{ca^2+a^2x}$ to its lowest terms.

$$cx+x^2)ca^2+a^2x$$
$$or\ c+x\ )ca^2+a^2x(a^2$$
$$ca^2+a^2x$$
$$*$$

Therefore $c+x$ *is the greatest common measure,* and $c+x)\frac{cx+x^2}{ca^2+a^2x}=\frac{x}{a^2}=$ *fraction required.*

2. Having $\frac{x^3-b^2x}{x^2+2bx+b^2}$ given, it is required to reduce it to its least terms.

$$x^2+2bx+b^2)x^3-b^2x(x$$
$$x^3+2bx^2+b^2x$$
$$-2bx^2-2b^2x)x^2+2bx+b^2$$
$$or\ x+b)x^2+2bx+b^2(x+b$$
$$x^2+\ bx$$
$$bx+b^2$$
$$bx+b^2$$
$$*$$

Therefore $x+b$ *is the greatest common measure,*

and $x+b)\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x^2-bx}{a+b}=$ *fraction required.*

3. To reduce $\frac{x^4-b^4}{x^5-b^2x^3}$ to its lowest terms.

Ans. $\frac{x^2+b^2}{x^3}$.

4. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

Ans. $\frac{a^2+x^2}{a-x}$.

PROBLEM VI.

To add fractional quantities together.

RULE.

1. Reduce the fractions to a common denominator, as in problem the third.

2. Add all the numerators together, and under their sum write the common denominator, and it will give the sum of the fractions required.

EXAMPLES:

1. Having $\frac{x}{2}$ *and* $\frac{x}{3}$ given, to find their sum.

$$x\times 3 = 3x$$
$$x\times 2 = 2x$$

$$2\times 3 = 6$$

$\frac{3x}{6}+\frac{2x}{6}=\frac{5x}{6}=$ *sum required.*

2. Having $\frac{a}{b}$, $\frac{c}{d}$ *and* $\frac{e}{f}$ given, to find their sum.

$$a \times d \times f = adf$$
$$c \times b \times f = cbf$$
$$e \times b \times d = ebd$$

$$b \times d \times f = bdf$$

$$\frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf} = \textit{sum required.}$$

3. Add $a - \frac{3x^2}{b}$ *and* $b + \frac{x - 2b}{c}$ together.

$$3x^2 \times c = 3cx^2$$
$$x - 2b \times b = bx - 2b^2$$

$$b \times c = bc$$

$$a - \frac{3cx^2}{bc}$$
$$b + \frac{bx - 2b^2}{bc}$$

$$a + b + \frac{bx - 3cx^2 - 2b^2}{bc} = \textit{sum required.}$$

4. Add $\frac{3x}{2b}$ *and* $\frac{x}{5}$ together. *Sum* $\frac{15x + 2bx}{10b}$

5. Add $\frac{x}{2}$, $\frac{x}{3}$ *and* $\frac{x}{4}$ together. *Sum* $x + \frac{x}{12}$

6. Add $\frac{x - 2}{3}$ *and* $\frac{4x}{7}$ together. *Sum* $\frac{19x - 14}{21}$

7. Add $x + \frac{x - 2}{3}$ *and* $3x + \frac{2x - 3}{4}$ together.

Sum. $4x + \frac{10x - 17}{12}$

PROBLEM VII.

To subtract one fractional quantity from another.

RULE.

1. Reduce the fractions to a common denominator, as in addition.

2. Subtract the numerators from each other, and under their difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES:

1. To find the difference of $\frac{x}{3}$ and $\frac{2x}{11}$.

$$x \times 11 = 11x$$
$$2x \times 3 = 6x$$
$$\overline{}$$
$$3 \times 11 = 33$$

$$\frac{11x}{33} - \frac{6x}{33} = \frac{5x}{33} = \textit{difference required.}$$

2. To find the difference of $\frac{x-a}{3b}$ and $\frac{2a-4x}{5c}$.

$$\overline{x-a} \times 5c = 5cx - 5ac$$
$$2a-4x \times 3b = 6ab - 12bx$$
$$\overline{}$$
$$3b \times 5c = 15bc$$

$$\frac{5cx-5ac}{15bc} - \frac{6ab-12bx}{15bc} = \frac{5cx-5ac-6ab+12bx}{15bc} =$$ *difference required.*

3. From $\frac{3x}{7}$ take $\frac{2x}{9}$. *Difference* $\frac{13x}{63}$.

4. From $\frac{x+a}{b}$ take $\frac{c}{d}$. *Difference* $\frac{dx+ad-bc}{bd}$.

5. From $\frac{3x+a}{5b}$ take $\frac{2x+7}{8}$.

Difference $\frac{24x+8a-10bx-35b}{40b}$.

6. From $3x+\frac{x^2}{b}$ take $x-\frac{x-a}{c}$.

Difference $2x+\frac{cx^2-bx+ab}{bc}$.

PROBLEM VIII.

To multiply fractional quantities together.

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator, and it will give the product required.

EXAMPLES:

1. Find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

$$\left.\begin{matrix} x\times 2x \\ 6\times 9 \end{matrix}\right\} = \frac{2x^2}{54} = \frac{x^2}{27} = \textit{product required.}$$

2. Find the product of $\frac{x}{2}$, $\frac{4x}{5}$ and $\frac{10x}{21}$.

$$\left.\begin{matrix} x\times 4x\times 10x \\ 2\times 5\times 21\end{matrix}\right\}=\frac{40x^3}{210}=\frac{4x^3}{21}=\textit{product required.}$$

3. Find the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$.

$$\left.\begin{matrix} x\times\overline{x+a} \\ a\times\overline{a+c}\end{matrix}\right\}=\frac{x^2+ax}{a^2+ac}=\textit{product required.}$$

4. Find the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. *Product* $\frac{9ax}{2b}$.

5. Find the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. *Product* $\frac{3x^3}{5a}$.

6. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$ and $\frac{3ac}{2b}$. *Product* $9xa$.

7. Find the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$.

Product $\frac{ab+bx}{x}$.

8. Find the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

Product $\frac{x^4-b^4}{cb^2-bc^2}$.

PROBLEM IX.

To divide one fractional quantity by another.

RULE.

Invert the divisor, and proceed as in multiplication.

EXAMPLES:

1. Find the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$.

$$\frac{x}{3}\times\frac{9}{2x}=\frac{9x}{6x}=\frac{3}{2}=1\tfrac{1}{2}=\textit{quotient required.}$$

2. Find the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$.

$$\frac{2a}{b}\times\frac{d}{4c}=\frac{2ad}{4bc}=\frac{ad}{2bc}=\textit{quotient required.}$$

3. Find the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

$$\frac{x+a}{2x-2b}\times\frac{5x+a}{x+b}=\frac{5x^2+6ax+a^2}{2x^2-2b^2}=\textit{quotient requir.}$$

4. Divide $\frac{5x}{3}$ by $\frac{2a}{3b}$. *Quotient* $\frac{5bx}{a}$.

5. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$. *Quotient* $\frac{x-b}{6c^2x}$.

6. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

Quotient $x+\frac{b^2}{x}$.

INVOLUTION.

Involution is the raising of powers from any proposed root; or the method of finding the square, cube, biquadrate, &c. of any given quantity.

RULE.

Multiply the quantity into itself as often as is denoted by the index, and the last product will be the power required. *Or*

Multiply the index of the quantity by the index of the power, and the result will be the same as before.

Note, When the sign of the root is + all the powers of it will be +; and when the sign is — all the odd powers will be —, and all the even powers +.

EXAMPLES:

$$a, \text{ root} \left\{ \begin{array}{l} a^2 = \text{square} \\ a^3 = \text{cube} \\ a^4 = \text{4th power} \\ a^5 = \text{5th power} \\ \quad \&c. \end{array} \right. \qquad a^2, \text{ root} \left\{ \begin{array}{l} a^4 = \text{square} \\ a^6 = \text{cube} \\ a^8 = \text{4th power} \\ a^{10} = \text{5th power} \\ \quad \&c. \end{array} \right.$$

$$-3a, \text{ root} \left\{ \begin{array}{l} +\ 9a^2 = \text{square} \\ -\ 27a^3 = \text{cube} \\ +\ 81a^4 = \text{4th power} \\ -243a^5 = \text{5th power} \end{array} \right.$$

$$-2ax^2, \text{ root} \left\{ \begin{array}{l} +\ 4a^2x^4 = \text{square} \\ -\ 8a^3x^6 = \text{cube} \\ +16a^4x^8 = \text{4th power} \\ -32a^5x^{10} = \text{5th power} \end{array} \right.$$

$$\frac{x}{a}, \text{ root} \left\{ \begin{array}{l} \frac{x^2}{a^2} = \text{square} \\ \frac{x^3}{a^3} = \text{cube} \\ \frac{x^4}{a^4} = \text{biquadrate} \end{array} \right.$$

$$-\frac{2ax^2}{3b},\ root \left\{ \begin{array}{l} +\dfrac{4a^2x^4}{9b^2} = square \\ -\dfrac{8a^3x^6}{27b^3} = cube \\ +\dfrac{16a^4x^8}{81b^4} = 4th\ power \end{array} \right.$$

$$\begin{array}{l} x + a = root \\ x + a \\ \hline x^2 + ax \\ \quad + ax + a^2 \\ \hline x^2 + 2ax + a^2 = square \\ x + a \\ \hline x^3 + 2ax^2 + a^2x \\ \quad + ax^2 + 2a^2x + a^3 \\ \hline x^3 + 3ax^2 + 3a^2x + a^3 = cube \\ x + a \\ \hline x^4 + 3ax^3 + 3a^2x^2 + a^3x \\ \quad + ax^3 + 3a^2x^2 + 3a^3x + a^4 \\ \hline x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4 = 4th\ power. \end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the cube or third power of $2a^2$.
Ans. $8a^6$.
2. Required the 4th power of $2a^2x$. *Ans.* $16a^8x^4$.
3. To find the 3d power of $-8x^2y^3$.
Ans. $512x^6y^9$.

4. To find the biquadrate of $-\frac{2a^2x}{3b^2}$.

Ans. $\frac{16a^8x^4}{81b^8}$.

5. Required the 5th power of $a-x$.

Ans. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

SIR ISAAC NEWTON'S RULE *for raising a binomial or residual quantity to any power whatever.*

1. *To find the terms without the co-efficients.* The index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and in the following quantity the indices of the terms are 0, 1, 2, 3, 4, &c.

2. *To find the unciæ or co-efficients.* The first is always 1, and the second is the index of the power: and in general, if the co-efficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the co-efficient of the term next following.

Note, The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the second term be −, then all the odd terms will be +, and the even terms −.

EXAMPLES:

1. Let $a+x$ be involved to the fifth power.

The terms without the co-efficients will be

a^5, a^4x, a^3x^2, a^2x^3, ax^4, x^5,

and the co-efficients will be

$$1,\ 5,\ \frac{5\times4}{2},\ \frac{10\times3}{3},\ \frac{10\times2}{4},\ \frac{5\times1}{5};$$

or 1, 5, 10, 10, 5, 1,

And therefore the 5th power is

$$a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

2. Let $x-a$ be involved to the 6th power.

The terms without the co-efficients will be

$$x^6,\ x^5a,\ x^4a^2,\ x^3a^3,\ x^2a^4,\ xa^5,\ a^6,$$

and the co-efficients will be

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6},$$

or 1, 6, 15, 20, 15, 6, 1,

And therefore the sixth power of $x-a$ is

$$x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6.$$

3. Find the 4th power of $x-a$.

Ans. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.

4. Find the 7th power of $x+b$.

Ans. $x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7$.

EVOLUTION.

Evolution is the reverse of involution, and teaches to find the roots of any given powers.

CASE I.

To find the roots of simple quantities.

RULE.

Extract the root of the co-efficient for the numerical part, and divide the index of the letters by the index of the power, and it will give the root required.

EXAMPLES:

1. The square root of $9x^2 = 3x^{\frac{2}{2}} = 3x$.
2. The cube root of $8x^3 = 2x^{\frac{3}{3}} = 2x$.
3. The square root of $3a^2x^6 = a^{\frac{2}{2}}x^{\frac{6}{2}}\sqrt{3} = ax^3\sqrt{3}$.
4. The cube root of $-125a^3x^6 = 5a^{\frac{3}{3}}x^{\frac{6}{3}} = 5ax^2$.
5. The biquadrate root of $16a^4x^8 = 2a^{\frac{4}{4}}x^{\frac{8}{4}} = 2ax^2$.

CASE II.

To find the square root of a compound quantity.

RULE.

1. Range the quantities according to the dimensions of some letter, and set the root of the first term in the quotient.

2. Subtract the square of the root, thus found, from the first term, and bring down the two next terms to the remainder for a dividend.

3. Divide the dividend by double the root, and set the result in the quotient.

4. Multiply the divisor and quotient by the term last put in the quotient, and subtract the product from the dividend, and so on, as in common arithmetic.

EXAMPLES:

1. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

$$4a^4+12a^3x+13a^2x^2+6ax^3+x^4(2a^2+3ax+x^2$$
$$4a^4$$

$$4a^2+3ax)\ 12a^3x+13a^2x^2$$
$$12a^3x+9a^2x^2$$

$$4a^2+6ax+x^2)\ 4a^2x^2+6ax^3+x^4$$
$$4a^2x^2+6ax^3+x^4$$

$$*$$

2. Extract the square root of $x^4-4x^3+6x^2-4x+1$.

$$x^4-4x^3+6x^2-4x+1(x^2-2x+1$$
$$x^4$$

$$2x^2-2x)-4x^3+6x^2$$
$$-4x^3+4x^2$$

$$2x^2-4x+1)\ 2x^2-4x+1$$
$$2x^2-4x+1$$

$$*$$

3. Required the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$. *Anf.* $a^2+2ax+x^2$.

4. Required the square root of $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$. *Anf.* $x^2-x+\frac{1}{4}$.

5. Required the square root of a^2+x^2.

Anf. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$ &c.

CASE III.

To find the roots of powers in general.

RULE.

1. Find the root of the first term, and place it in the quotient.

2. Subtract the power, and bring down the second term for a dividend.

3. Involve the root, last found, to the next lowest power, and multiply it by the index of the given power for a divisor.

4. Divide the dividend by the divisor, and the quotient will be the next term of the root.

5. Involve the whole root, and subtract and divide as before; and so on till the whole is finished.

EXAMPLES:

1. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4(a^2-ax+x^2$$
$$a^4$$
$$\overline{2a^2)-2a^3x}$$
$$\overline{a^4-2a^3x+\ a^2x^2}$$
$$\overline{2a^2)2a^2x^2}$$
$$\overline{a^4-2a^3x+3a^2x^2-2ax^3+x^4}$$

*

2. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$\begin{array}{l}
x^6+6x^5-40x^3+96x-64(x^2+2x-4 \\
x^6 \\
\hline
3x^4)\ 6x^5 \\
\hline
x^6+6x^5+12x^4+8x^3 \\
\hline
\quad 3x^4)-12x^4 \\
\hline
x^6+6x^5-40x^3+96x-64 \\
\hline
\end{array}$$

*

3. Required the square root of $a^2+2ab+2ac+b^2+2bc+c^2$. *Ans.* $a+b+c$.

4. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. *Ans.* x^2-2x+1.

5. Required the biquadrate root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$. *Ans.* $2a-3x$.

SURDS.

Surds are such quantities as have no exact root, and are usually expressed by fractional indices: thus, the square root of 2, and the cube root of 3, &c. cannot be exactly determined, but may be denoted by $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ &c.

PROBLEM I.

To reduce a rational quantity to the form of a Surd.

RULE.

Multiply the index of the rational quantity by the index of the ſurd, and over this new quantity place the radical ſign, and it will be of the form required.

EXAMPLES:

1. 3 reduced to the form of $\sqrt{7}$ is $\sqrt{3^{1\times 2}}=\sqrt{3^2}=\sqrt{9}$.
2. x reduced to the form of $\sqrt[3]{a}$ is $\sqrt[3]{x^{1\times 3}}=\sqrt[3]{x^3}$.
3. $a+b$ reduced to the form of $\sqrt{cx}$ is $\sqrt{\overline{a+b}|^2}=\sqrt{a^2+2ab+b^2}$.
4. $\frac{a}{b\sqrt{x}}$ reduced to the form of $\sqrt{c}$ is $\sqrt{\frac{a^2}{b^2x}}$.

PROBLEM II.

To reduce quantities of different indices to other equivalent ones, that ſhall have a common index.

RULE.

1. Divide the indices of the quantities by the given index, and the quotients will be the new indices for thoſe quantities.
2. Over the ſaid quantities, with their new indices, place the given index, and they will make the equivalent quantities required.

EXAMPLES:

1. Reduce $15^{\frac{1}{4}}$ and $9^{\frac{1}{6}}$ to equivalent quantities having the common index $\frac{1}{2}$.

$$\tfrac{1}{4} \div \tfrac{1}{2} = \tfrac{1}{4} \times \tfrac{2}{1} = \tfrac{2}{4} = \tfrac{1}{2} = \textit{1st index.}$$
$$\tfrac{1}{6} \div \tfrac{1}{2} = \tfrac{1}{6} \times \tfrac{2}{1} = \tfrac{2}{6} = \tfrac{1}{3} = \textit{2d index}$$

Therefore $\overline{15^{\frac{1}{2}}}|^{\frac{1}{2}}$ and $\overline{9^{\frac{1}{3}}}|^{\frac{1}{2}}$ = *quantities required.*

2. Reduce a^2 and $x^{\frac{1}{4}}$ to the same common index $\frac{1}{3}$.

$$\tfrac{2}{1} \div \tfrac{1}{3} = \tfrac{2}{1} \times \tfrac{3}{1} = \tfrac{6}{1} = \textit{1st index}$$
$$\tfrac{1}{4} \div \tfrac{1}{3} = \tfrac{1}{4} \times \tfrac{3}{1} = \tfrac{3}{4} = \textit{2d index}$$

Therefore $\overline{a^6}|^{\frac{1}{3}}$ and $\overline{x^{\frac{3}{4}}}|^{\frac{1}{3}}$ = *quantities required.*

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Ans. $\overline{27}|^{\frac{1}{6}}$ *and* $\overline{4}|^{\frac{1}{6}}$

4. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

Ans. $\overline{a^{\frac{8}{3}}}|^{\frac{1}{8}}$ *and* $\overline{b^2}|^{\frac{1}{8}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to the same radical sign.

Ans. $\sqrt[nm]{a^m}$ *and* $\sqrt[nm]{b^n}$.

PROBLEM III.

To reduce surds to their most simple terms.

RULE.

Find the root of the greatest power contained in the given surd, and set it before the remaining quantities, with the proper radical sign between them.

EXAMPLES:

1. $\sqrt{48} = \sqrt{3 \times 16} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$.
2. $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$.

3. $\sqrt[3]{81}=\sqrt[3]{27\times 3}=\sqrt[3]{27}\times\sqrt[3]{3}=3\sqrt[3]{3}$.

4. $\sqrt{49a^2x}=\sqrt{49a^2\times x}=\sqrt{49a^2}\times\sqrt{x}=7a\sqrt{x}$.

5. $\overline{bx+x^2}|^{\frac{1}{2}}=x^{\frac{1}{2}}\times\overline{b+x}|^{\frac{1}{2}}$; $\overline{a^3x-a^2x^2}|^{\frac{1}{2}}=a\times\overline{ax-x^2}|^{\frac{1}{2}}$.

6. $\overline{\frac{64a^4x^3}{27a-27x}}|^{\frac{1}{3}}=\frac{4ax}{3}\times\overline{\frac{a}{a-x}}|^{\frac{1}{3}}$.

PROBLEM IV.

To add surd quantities together.

RULE.

1. Reduce those quantities that have unlike indices to equivalent ones, having a common index.

2. Bring all fractions to a common denominator, and reduce the quantities to their simplest terms, as in the last problem.

3. Then, if the surd part be the same in them all, annex it to the sum of the rational parts, with the sign of multiplication, and it will give the total sum required.

If the surd part be not the same in all the quantities, they can only be added by the signs $+$ and $-$.

EXAMPLES:

1. $2\sqrt{5}+7\sqrt{5}=\overline{2+7}\times\sqrt{5}=9\sqrt{5}$.

2. $\sqrt{27}+\sqrt{48}=3\sqrt{3}+4\sqrt{3}=7\sqrt{3}$.

3. $\overline{500}|^{\frac{1}{3}}+\overline{108}|^{\frac{1}{3}}=\overline{125\times4}|^{\frac{1}{3}}+\overline{27\times4}|^{\frac{1}{3}}=5\times4^{\frac{1}{3}}$ $+3\times4^{\frac{1}{3}}=8\times4^{\frac{1}{3}}$.

4. $\sqrt{27a^4x}+\sqrt{3a^2x^3}=\sqrt{9a^4\times3x}+\sqrt{a^2x^2\times3x}$ $=3a^2\sqrt{3x}+ax\sqrt{3x}=\overline{3a+x}\times a\times\sqrt{3x}$.

5. $\sqrt{\frac{24}{25}}+\sqrt{\frac{2}{3}}=\sqrt{\frac{72}{75}}+\sqrt{\frac{50}{75}}=\sqrt{\frac{36\times2}{75}}+$ $\sqrt{\frac{25\times2}{75}}=6\sqrt{\frac{2}{75}}+5\sqrt{\frac{2}{75}}=11\sqrt{\frac{2}{75}}$.

6. $\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{64}{108}}=\sqrt[3]{\frac{1}{8\times4}}+\sqrt[3]{\frac{64\times1}{27\times4}}=\frac{1}{2}$ $\times\sqrt[3]{\frac{1}{4}}+\frac{4}{3}\times\sqrt[3]{\frac{1}{4}}=\overline{\frac{1}{2}+\frac{4}{3}}\times\sqrt[3]{\frac{1}{4}}=\frac{11}{6}\times$ $\sqrt[3]{\frac{1}{4}}$.

7. $\sqrt{\frac{a}{b}}+\sqrt{\frac{b^3}{a^3}}=\sqrt{\frac{a^4}{ba^3}}+\sqrt{\frac{b^4}{ba^3}}=\frac{a^2}{a}\times\sqrt{\frac{1}{ba}}+$ $\frac{b^2}{a}\times\sqrt{\frac{1}{ba}}=\frac{a^2+b^2}{a}\times\sqrt{\frac{1}{ba}}=\frac{a^2+b^2}{a\sqrt{ba}}$.

PROBLEM V.

To ſubtract ſurd quantities.

RULE.

Prepare the quantities as in the laſt rule, and annex the difference of the rational parts to the common ſurd, with the ſign of multiplication.

EXAMPLES:

1. $3\sqrt{18}-\sqrt{18}=\overline{3-1}\times\sqrt{18}=2\sqrt{18}$.

2. $2\sqrt{50}-\sqrt{18}=2\sqrt{\overline{25\times2}}-\sqrt{\overline{9\times2}}=2\times5\times\sqrt{2}-1\times3\times\sqrt{2}=7\sqrt{2}$.

3. $\sqrt[3]{192}-\sqrt[3]{24}=\sqrt[3]{\overline{64\times3}}-\sqrt[3]{\overline{8\times3}}=4\times3^{\frac{1}{3}}-2\times3^{\frac{1}{3}}=\overline{4-2}\times3^{\frac{1}{3}}=2\times3^{\frac{1}{3}}$.

4. $\sqrt{80a^4x}-\sqrt{20a^2x^3}=\sqrt{\overline{16a^4\times5x}}-\sqrt{\overline{4a^2x^2\times5x}}=4a^2\sqrt{5x}-2ax\sqrt{5x}=\overline{4a^2-2ax}\times\sqrt{5x}$.

5. $\sqrt{\frac{24}{25}}-\sqrt{\frac{2}{3}}=\sqrt{\frac{72}{75}}-\sqrt{\frac{50}{75}}=\sqrt{\frac{36\times2}{75}}-\sqrt{\frac{25\times2}{75}}=6\times\sqrt{\frac{2}{75}}-5\times\sqrt{\frac{2}{75}}=\overline{6-5}\times\sqrt{\frac{2}{75}}=\sqrt{\frac{2}{75}}$.

6. $\sqrt[3]{\frac{16}{27}}-\sqrt[3]{\frac{1}{4}}=\frac{4}{3}\times\sqrt[3]{\frac{1}{4}}-\sqrt[3]{\frac{1}{4}}=\overline{\frac{4}{3}-\frac{3}{3}}\times\sqrt[3]{\frac{1}{4}}=\frac{1}{3}\times\sqrt[3]{\frac{1}{4}}$.

7. $\sqrt{\frac{b}{c}}-\sqrt{\frac{c^3}{b^3}}=\sqrt{\frac{b^4}{cb^3}}-\sqrt{\frac{c^4}{cb^3}}=\frac{b^2}{b}\times\sqrt{\frac{1}{cb}}-\frac{c^2}{b}\times\sqrt{\frac{1}{cb}}=\frac{b^2-c^2}{b}\times\sqrt{\frac{1}{cb}}=\frac{b^2-c^2}{b\times\overline{cb}|^{\frac{1}{2}}}$.

PROBLEM VI.

To multiply surd quantities together.

RULE.

Reduce the surds to the same index; and then the product of the rational quantities being annexed to

the product of the surds, will give the whole product required.

EXAMPLES:

1. $\sqrt{10} \times \sqrt{5} = \sqrt{50} = \sqrt{\overline{25 \times 2}} = 5\sqrt{2}$.

2. $3\sqrt{8} \times 2\sqrt{6} = 3 \times 2\sqrt{\overline{8 \times 6}} = 6\sqrt{48} = 6\sqrt{\overline{16 \times 3}}$
$= 4 \times 6 \times \sqrt{3} = 24\sqrt{3}$.

3. $\frac{2}{3}\sqrt{\frac{1}{8}} \times \frac{3}{4}\sqrt{\frac{7}{10}} = \frac{2}{3}\sqrt{\frac{8}{64}} \times \frac{3}{4}\sqrt{\frac{7}{10}} = \frac{6}{12}$
$\sqrt{\frac{56}{640}} = \frac{1}{2}\sqrt{\frac{4 \times 14}{64 \times 10}} = \frac{1}{2} \times \frac{2}{8} \times \sqrt{\frac{14}{10}} = \frac{2}{16}\sqrt{1}$
$\frac{4}{10} = \frac{1}{8}\sqrt{1\frac{2}{5}}$.

4. $\frac{1}{3} \times \sqrt[3]{\frac{3}{2}} \times \frac{1}{5} \times \sqrt[3]{\frac{1}{4}} = \frac{1}{3} \times \sqrt[3]{\frac{3}{2}} \times$
$\frac{1}{5} \times \sqrt[3]{\frac{3}{12}} = \frac{1}{3} \times \frac{1}{5} \times \sqrt[3]{\frac{3 \times 3}{2 \times 12}} = \frac{1}{15} \times \sqrt[3]{\frac{9}{24}}$
$= \frac{1}{15} \times \sqrt[3]{\frac{3}{8}} = \frac{1}{15} \times \frac{1}{2} \times 3^{\frac{1}{3}} = \frac{1}{30} \times 3^{\frac{1}{3}}$.

5. $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{3}{6}} \times a^{\frac{2}{6}} = a^{\frac{5}{6}} = \overline{a^5}|^{\frac{1}{6}}$.

6. $5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 6x^{\frac{2}{3}}y^{\frac{3}{4}} = 5x^{\frac{6}{12}}y^{\frac{4}{12}} \times 6x^{\frac{8}{12}}y^{\frac{9}{12}} = 30x^{\frac{14}{12}}$
$y^{\frac{13}{12}} = 30 \times \overline{x^{14}y^{13}}|^{\frac{1}{12}}$.

7. $\overline{x+y}|^{\frac{1}{2}} \times \overline{x+y}|^{\frac{1}{3}} = \overline{x+y}|^{\frac{3}{6}} \times \overline{x+y}|^{\frac{2}{6}}$
$= \overline{\overline{x+y}^{3}}|^{\frac{1}{6}} \times \overline{\overline{x+y}^{2}}|^{\frac{1}{6}} = \overline{\overline{x+y}^{5}}|^{\frac{1}{6}}$.

2. $2\sqrt{50}-\sqrt{18}=2\sqrt{\overline{25\times2}}-\sqrt{\overline{9\times2}}=2\times5\times\sqrt{2}-1\times3\times\sqrt{2}=7\sqrt{2}$.

3. $\sqrt[3]{192}-\sqrt[3]{24}=\sqrt[3]{\overline{64\times3}}-\sqrt[3]{\overline{8\times3}}=4\times3^{\frac{1}{3}}-2\times3^{\frac{1}{3}}=\overline{4-2}\times3^{\frac{1}{3}}=2\times3^{\frac{1}{3}}$.

4. $\sqrt{80a^4x}-\sqrt{20a^2x^3}=\sqrt{\overline{10a^4\times5x}}-\sqrt{\overline{4a^2x^2\times5x}}=4a^2\sqrt{5x}-2ax\sqrt{5x}=\overline{4a^2-2ax}\times\sqrt{5x}$.

5. $\sqrt{\frac{24}{25}}-\sqrt{\frac{2}{3}}=\sqrt{\frac{72}{75}}-\sqrt{\frac{50}{75}}=\sqrt{\frac{36\times2}{75}}-\sqrt{\frac{25\times2}{75}}=6\times\sqrt{\frac{2}{75}}-5\times\sqrt{\frac{2}{75}}=\overline{6-5}\times\sqrt{\frac{2}{75}}=\sqrt{\frac{2}{75}}$.

6. $\sqrt[3]{\frac{16}{27}}-\sqrt[3]{\frac{1}{4}}=\frac{4}{3}\times\sqrt[3]{\frac{1}{4}}-\sqrt[3]{\frac{1}{4}}=\overline{\frac{4}{3}-\frac{3}{3}}\times\sqrt[3]{\frac{1}{4}}=\frac{1}{3}\times\sqrt[3]{\frac{1}{4}}$.

7. $\sqrt{\frac{b}{c}}-\sqrt{\frac{c^3}{b^3}}=\sqrt{\frac{b^4}{cb^3}}-\sqrt{\frac{c^4}{cb^3}}=\frac{b^2}{b}\times\sqrt{\frac{1}{cb}}-\frac{c^2}{b}\times\sqrt{\frac{1}{cb}}=\frac{b^2-c^2}{b}\times\sqrt{\frac{1}{cb}}=\frac{b^2-c^2}{b\times\overline{cb}|^{\frac{1}{2}}}$.

PROBLEM VI.

To multiply surd quantities together.

RULE.

Reduce the surds to the same index; and then the product of the rational quantities being annexed to

the product of the surds, will give the whole product required.

EXAMPLES:

1. $\sqrt{10}\times\sqrt{5}=\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}$.

2. $3\sqrt{8}\times 2\sqrt{6}=3\times 2\sqrt{8\times 6}=6\sqrt{48}=6\sqrt{16\times 3}=4\times 6\times\sqrt{3}=24\sqrt{3}$.

3. $\frac{2}{3}\sqrt{\frac{1}{8}}\times\frac{3}{4}\sqrt{\frac{7}{10}}=\frac{2}{3}\sqrt{\frac{8}{64}}\times\frac{3}{4}\sqrt{\frac{7}{10}}=\frac{6}{12}\sqrt{\frac{56}{640}}=\frac{1}{2}\sqrt{\frac{4\times 14}{64\times 10}}=\frac{1}{2}\times\frac{2}{8}\times\sqrt{\frac{14}{10}}=\frac{2}{16}\sqrt{1\frac{4}{10}}=\frac{1}{8}\sqrt{1\frac{2}{5}}$.

4. $\frac{1}{3}\times\sqrt[3]{\frac{3}{2}}\times\frac{1}{5}\times\sqrt[3]{\frac{1}{4}}=\frac{1}{3}\times\sqrt[3]{\frac{3}{2}}\times\frac{1}{5}\times\sqrt[3]{\frac{3}{12}}=\frac{1}{3}\times\frac{1}{5}\times\sqrt[3]{\frac{3\times 3}{2\times 12}}=\frac{1}{15}\times\sqrt[3]{\frac{9}{24}}=\frac{1}{15}\times\sqrt[3]{\frac{3}{8}}=\frac{1}{15}\times\frac{1}{2}\times 3^{\frac{1}{3}}=\frac{1}{30}\times 3^{\frac{1}{3}}$.

5. $a^{\frac{1}{2}}\times a^{\frac{1}{3}}=a^{\frac{3}{6}}\times a^{\frac{2}{6}}=a^{\frac{5}{6}}=\overline{a^5}|^{\frac{1}{6}}$.

6. $5x^{\frac{1}{2}}y^{\frac{1}{3}}\times 6x^{\frac{2}{3}}y^{\frac{3}{4}}=5x^{\frac{6}{12}}y^{\frac{4}{12}}\times 6x^{\frac{8}{12}}y^{\frac{9}{12}}=30x^{\frac{14}{12}}y^{\frac{13}{12}}=30\times\overline{x^{14}y^{13}}|^{\frac{1}{12}}$.

7. $\overline{x+y}|^{\frac{1}{2}}\times\overline{x+y}|^{\frac{1}{3}}=\overline{x+y}|^{\frac{3}{6}}\times\overline{x+y}|^{\frac{2}{6}}=\overline{\overline{x+y}^3}|^{\frac{1}{6}}\times\overline{\overline{x+y}^2}|^{\frac{1}{6}}=\overline{\overline{x+y}^5}|^{\frac{1}{6}}$.

8. $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} \times \frac{x^{\frac{1}{3}}}{y^{\frac{1}{2}}} = \frac{x^{\frac{3}{6}}}{y^{\frac{2}{6}}} \times \frac{x^{\frac{2}{6}}}{y^{\frac{3}{6}}} = \overline{\frac{x^3}{y^2}}\Big|^{\frac{1}{6}} \times \overline{\frac{x^2}{y^3}}\Big|^{\frac{1}{6}} = \overline{\frac{x^5}{y^5}}\Big|^{\frac{1}{6}}.$

9. $x^{\frac{1}{n}} \times y^{\frac{1}{m}} = x^{\frac{m}{nm}} \times y^{\frac{n}{nm}} = \overline{x^m}\Big|^{\frac{1}{nm}} \times \overline{y^n}\Big|^{\frac{1}{nm}} = \overline{x^m y^n}\Big|^{\frac{1}{nm}}.$

10. $\overline{x+\sqrt{y}} \times \overline{x-\sqrt{y}} = x^2 - y.$

11. $\overline{x-x\sqrt{a}} \times \overline{y-y\sqrt{a}} = xy - 2xy\sqrt{a} + axy.$

12. $\sqrt{a+\sqrt{b-\sqrt{3}}} \times \sqrt{a-\sqrt{b-\sqrt{3}}}$
$= \sqrt{a^2-b+\sqrt{3}}.$

PROBLEM VII.

To divide one surd quantity by another.

RULE.

Reduce the surds to the same index; and then the quotient of the rational quantities being annexed to the quotient of the surds, will give the whole quotient required.

EXAMPLES:

1. $\sqrt{375} \div \sqrt{5} = \sqrt{\frac{375}{5}} = \sqrt{75} = \sqrt{\overline{25 \times 3}} = 5\sqrt{3}.$

2. $8\sqrt{108} \div 2\sqrt{6} = \frac{8}{2}\sqrt{\frac{108}{6}} = 4\sqrt{18} = 4\sqrt{\overline{9 \times 2}}$
$= 12\sqrt{2}.$

3. $\frac{3}{4}\sqrt{\frac{1}{135}} \div \frac{2}{3}\sqrt{\frac{1}{5}} = \frac{3}{4}\times\frac{3}{2}\times\sqrt{\frac{1}{135}\times\frac{5}{1}}$
$= \frac{9}{8}\sqrt{\frac{5}{135}} = \frac{9}{8}\sqrt{\frac{1}{9\times3}} = \frac{3}{8}\sqrt{\frac{1}{3}}.$

4. $\frac{5}{7}\times\sqrt[3]{\frac{8}{189}} \div \frac{2}{5}\times\sqrt[3]{\frac{8}{7}} = \frac{5}{7}\times\frac{5}{2}\times$
$\left.\overline{\frac{8}{189}\times\frac{7}{8}}\right|^{\frac{1}{3}} = \frac{25}{42}\times\left.\overline{\frac{1}{3}}\right|^{\frac{1}{3}}.$

5. $x^{\frac{1}{2}}\div y^{\frac{1}{3}} = x^{\frac{3}{6}}\div y^{\frac{2}{6}} = \left.\overline{x^3}\right|^{\frac{1}{6}}\div\left.\overline{y^2}\right|^{\frac{1}{6}} = \left.\overline{\frac{x^3}{y^2}}\right|^{\frac{1}{6}}.$

6. $\frac{2}{5}x^{\frac{1}{2}}y^{\frac{1}{2}}\div\frac{3}{4}x^{\frac{1}{3}}y^{\frac{1}{3}} = \frac{2}{5}x^{\frac{3}{6}}y^{\frac{3}{6}}\div\frac{3}{4}x^{\frac{2}{6}}y^{\frac{2}{6}} = \frac{2}{5}\times$
$\frac{4}{3}\times x^{\frac{1}{6}}y^{\frac{1}{6}} = \frac{8}{15}\times\left.\overline{xy}\right|^{\frac{1}{6}}.$

7. $\left.\overline{x+y}\right|^{\frac{1}{2}}\div\left.\overline{x+y}\right|^{\frac{1}{3}} = \left.\overline{x+y}\right|^{\frac{3}{6}}\div\left.\overline{x+y}\right|^{\frac{2}{6}} = \left.\overline{x+y}\right|^{\frac{1}{6}}.$

8. $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}}\div\frac{y^{\frac{1}{2}}}{x^{\frac{1}{3}}} = \frac{x^{\frac{3}{6}}}{y^{\frac{2}{6}}}\times\frac{x^{\frac{2}{6}}}{y^{\frac{3}{6}}} = \left.\overline{\frac{x^5}{y^5}}\right|^{\frac{1}{6}}.$

9. $x^{\frac{1}{n}}\div x^{\frac{1}{m}} = x^{\frac{m}{nm}}\div x^{\frac{n}{nm}} = x^{\frac{m-n}{nm}} = \left.\overline{x^{m-n}}\right|^{\frac{1}{nm}}$

10. $\overline{x^2-xd-b+d\sqrt{b}}\div\overline{x-\sqrt{b}} = x+\sqrt{b}-d.$

PROBLEM VIII.

To involve surd quantities to any power.

RULE.

Multiply the index of the quantity by the index of the power to be raised, and to the result annex the power of the rational parts, and it will be involved as required.

RULE.

Divide the numerator by the denominator, as in common divifion; and the operation continued, as far as fhall be thought neceffary, will give the feries required.

EXAMPLES:

1. Let $\frac{ax}{a-x}$ be propofed to be thrown into an infinite feries.

$$\begin{array}{l}
a-x)\,ax\ldots\left(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}\ \&c.\right.\\
\quad\ \ ax-x^2\\
\quad\ \ \overline{}\\
\qquad\quad x^2\\
\qquad\quad x^2-\frac{x^3}{a}\\
\qquad\quad \overline{}\\
\qquad\qquad\ \ \frac{x^3}{a}\\
\qquad\qquad\ \ \frac{x^3}{a}-\frac{x^4}{a^2}\\
\qquad\qquad\ \ \overline{}\\
\qquad\qquad\qquad \frac{x^4}{a^2}\\
\qquad\qquad\qquad \frac{x^4}{a^2}-\frac{x^5}{a^3}\\
\qquad\qquad\qquad \overline{}\\
\qquad\qquad\qquad\quad\ \ \frac{x^5}{a^3}\ \&c.
\end{array}$$

2. Let $\frac{a^2}{a^2+2ax+x^2}$ be propoſed to be thrown into an infinite ſeries.

$$\begin{array}{l} a^2+2ax+x^2)\,a^2 \quad . \quad . \quad . \quad (1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}\ \&c. \\ \qquad a^2+2ax+x^2 \\ \hline \qquad -2ax-x^2 \\ \qquad -2ax-4x^2-\frac{2x^3}{a} \\ \hline \qquad\qquad 3x^2+\frac{2x^3}{a} \\ \qquad\qquad 3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2} \\ \hline \qquad\qquad\quad -\frac{4x^3}{a}-\frac{3x^4}{a^2} \\ \qquad\qquad\quad -\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3} \\ \hline \qquad\qquad\qquad \frac{5x^4}{a^2}+\frac{4x^5}{a^3}\ \&c. \end{array}$$

3. To throw $\frac{b}{a+x}$ into an infinite ſeries.

Anſ. $\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+$ &c.

4. To throw $\frac{a^2}{x+b}$ into an infinite ſeries.

Anſ. $\frac{a^2}{x}-\frac{a^2b}{x^2}+\frac{a^2b^2}{x^3}-\frac{a^2b^3}{x^4}+$ &c.

5. To throw $\frac{1}{1+x^2}$ into an infinite ſeries.

Anſ. $1-x^2+x^4-x^6+x^8-$ &c.

6. To throw $\dfrac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ into an infinite series.

Ans. $2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^{2}+34x^{\frac{5}{2}}$ &c.

PROBLEM II.

To reduce a compound surd into an infinite series.

RULE.

Extract the root as in common arithmetic, and the operation, continued as far as shall be thought necessary, will give the series required.

EXAMPLES:

1. Extract the square root of a^2+x^2 in an infinite series.

$$
\begin{array}{rl}
 & a^2+x^2\left(a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}-\dfrac{5x^8}{128a^7}\right. \text{ \&c.} \\
 & a^2 \\
\hline
2a+\dfrac{x^2}{2a}\Big) & x^2 \\
 & x^2+\dfrac{x^4}{4a^2} \\
\hline
2a+\dfrac{x^2}{a}-\dfrac{x^4}{8a^3}\Big) & -\dfrac{x^4}{4a^2} \\
 & -\dfrac{x^4}{4a^2}-\dfrac{x^6}{8a^4}+\dfrac{x^8}{64a^6} \\
\hline
2a+\dfrac{x^2}{a}-\dfrac{x^4}{8a^3} \text{ \&c.}\Big) & \dfrac{x^6}{8a^4}-\dfrac{x^8}{64a^6} \\
 & \dfrac{x^6}{8a^4}+\dfrac{x^8}{16a^6} \text{ \&c.} \\
\hline
 & -\dfrac{5x^8}{64a^6} \text{ \&c.}
\end{array}
$$

2. Throw $\overline{x-x^2}|^{\frac{1}{2}}$ into an infinite series.

Ans. $x^{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{2}-\frac{x^{\frac{5}{2}}}{8}-\frac{x^{\frac{7}{2}}}{16}-\frac{5x^{\frac{9}{2}}}{128}$ &c.

3. Throw $\overline{a^2-x^2}|^{\frac{1}{2}}$ into an infinite series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^5}{16a^5}$ &c.

4. Throw $\overline{1-x^3}|^{\frac{1}{3}}$ into an infinite series.

Ans. $1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}$ &c.

PROBLEM III.

To reduce a binomial surd into an infinite series; or to extract any root of a binomial.

RULE.

Substitute the particular letters of the binomial, with their proper signs, in the following general form, and it will give the root required; observing that P is the first term, Q the second term divided by the first, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c. the foregoing terms, with their signs.

$$\overline{P+PQ}|^{\frac{m}{n}}=P^{\frac{m}{n}}(A)+\frac{m}{n}AQ(B)+\frac{m-n}{2n}BQ(C)+\frac{m-2n}{3n}CQ(D)+\frac{m-3n}{4n}DQ(E)\ \&c.$$

EXAMPLES.

1. To extract the square root of r^2-x^2, in an infinite series.

Here $P=r^2$, $Q=\frac{-x^2}{r^2}$, and $\frac{m}{n}=\frac{1}{2}$.

Therefore $\overline{r^2-x^2}|^{\frac{1}{2}}=r+\frac{1}{2}A\times\frac{-x^2}{r^2}-\frac{1}{4}B\times\frac{-x^2}{r^2}-\frac{3}{6}C\times\frac{-x^2}{r^2}-\frac{5}{8}D\times\frac{-x^2}{r^2}$ &c. $=r-\frac{x^2}{2r^2}A+\frac{x^2}{4r^2}B+\frac{3x^2}{6r^2}C+\frac{5x^2}{8r^2}D$ &c.

And, by restoring the values of A, B, C, D, &c. *we shall have* $\overline{r^2-x^2}|^{\frac{1}{2}}=r-\frac{x^2}{2r}-\frac{x^4}{8r^3}-\frac{x^6}{16r^5}-\frac{5x^8}{128r^7}$ &c. $=$ *series required.*

2. To extract the cube root of $\frac{a^2}{\overline{a^2+x^2}|^2}$ in an infinite series.

This expression, reduced, is $\frac{a^{\frac{2}{3}}}{\overline{a^2+x^2}|^{\frac{2}{3}}}=a^{\frac{2}{3}}\times\overline{a^2+x^2}|^{-\frac{2}{3}}$; *and therefore* $P=a^2$, $Q=\frac{x^2}{a^2}$, *and* $\frac{m}{n}=\frac{-2}{3}$.

Whence $\overline{a^2+x^2}|^{-\frac{2}{3}}=a^{2^{-\frac{2}{3}}}-\frac{2}{3}A\times\frac{x^2}{a^2}-\frac{5}{6}B\times\frac{x^2}{a^2}-\frac{8}{9}C\times\frac{x^2}{a^2}-\frac{11}{12}D\times\frac{x^2}{a^2}$ &c. $=\frac{1}{\sqrt[3]{a}}-\frac{2x^2}{3\sqrt[3]{a^7}}+\frac{5x^4}{9\sqrt[3]{a^{13}}}-\frac{40x^6}{81\sqrt[3]{a^{19}}}+\frac{110x^8}{243\sqrt[3]{a^{25}}}$ &c. $=\frac{1}{\sqrt[3]{a}}\times:\frac{1}{a}-\frac{2x^2}{3a^3}-\frac{5x^4}{9a^5}-\frac{40x^6}{81a^7}+\frac{110x^8}{243a^9}$ &c.

Therefore $\frac{a^{\frac{2}{3}}}{\overline{a^2+x^2}|^{\frac{2}{3}}}=\frac{1}{a^{\frac{1}{3}}}\times:1-\frac{2x^2}{3a^2}+\frac{5x^4}{9a^4}-\frac{40x^6}{81a^6}$ &c. *as required.*

3. To find the value of $\frac{r^2}{r+x}$, in an infinite series.

Ans. $r-x+\frac{x^2}{r}-\frac{x^3}{r^2}+\frac{x^4}{r^3}$ &c.

4. To find the value of $\frac{1}{\overline{a^2-x^2}|^{\frac{1}{2}}}$ in an infinite series.

Ans. $\frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}+\frac{15x^6}{48a^7}$ &c.

5. To find the value of $\frac{a^2}{\overline{a+x}|^2}$ in an infinite series.

Ans. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\frac{5x^4}{a^4}$ &c.

6. To find the value of $\frac{1}{\overline{2rx-x^2}|^{\frac{1}{2}}}$ in an infinite series.

Ans. $\frac{1}{\sqrt{2rx}}\times:1+\frac{x}{4r}+\frac{3x^2}{4.8r^2}+\frac{3.5x^3}{4.8.12r^3}$ &c.

7. Find the value of $\overline{a^2-x^2}|^{\frac{1}{5}}$, in an infinite series.

Ans. $a^{\frac{2}{5}}\times:1-\frac{x^2}{5a^2}-\frac{2x^4}{25a^4}-\frac{6x^6}{125a^6}$ &c.

8. To find the value of $\overline{1-x}|^{\frac{1}{4}}$ in an infinite series.

Ans. $1-\frac{x}{4}-\frac{3x^2}{4.8}-\frac{3.7x^3}{4.8.12}-\frac{3.7.11x^4}{4.8.12.16}$ &c.

9. To find the value of $\overline{\frac{a^2+x^2}{a^2-x^2}}|^{\frac{1}{2}}$ in a series.

Ans. $1+\frac{x^2}{a^2}+\frac{x^4}{2a^2}+\frac{x^6}{2a^6}$ &c.

10. To find the value of $\frac{ax}{a^2-ax+x^2}$ in a series.

Ans. $\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^4}{a^4}-\frac{x^5}{a^5}$ &c.

ARITHMETICAL PROPORTION.

Arithmetical proportion is the relation which two quantities, of the ſame kind, bear to each other, with reſpect to their difference.

Four quantities are ſaid to be in *arithmetical proportion*, when the difference between the firſt and ſecond is equal to the difference between the third and fourth.

Thus, 3, 7, 12, 16, *and* a, $a+b$, c, $c+b$, *are arithmetically proportional.*

Arithmetical progreſſion is when a ſeries of quantities either increaſe or decreaſe by the ſame common difference.

Thus, 2, 4, 6, 8, 10, 12, &c. *and* a, $a+b$, $a+2b$, $a+3b$, $a+4b$, $a+5b$, &c. *are ſeries in arithmetical progreſſion, whoſe common differences are* 2, *and* b.

The moſt uſeful part of arithmetical proportion is contained in the following theorems:

I. If four quantities be in arithmetical proportion, the ſum of the two means will be equal to the ſum of the two extremes.

Thus, if 2, 5, 7, 10, *and* a, b, c, d, *are in arithmetical proportion, then will* $2+10=5+7$, *and* $a+d=b+c$.

II. In any arithmetical ſeries, the laſt term is equal to the firſt, more the product of the common difference by the number of terms leſs one.

Thus the 20th term of 2, 4, 6, 8, 10, 12, &c. is $=2+\overline{20-1}\times 2=2+19\times 2=2+38=40$,

And the n*th term of* a, $a+x$, $a+2x$, $a+3x$, $a+4x$, &c. *is* $=a+\overline{n-1}\times x=a+\overline{n-1}x$.

III. The ſum of any ſeries of quantities in arithmetical progreſſion, is equal to the ſum of the two extremes multiplied by half the number of terms.

Thus the ſum of 1, 2, 3, 4, 5, 6, &c. *continued to the 20th term, is* $=\frac{\overline{1+20}\times 20}{2}=\frac{21\times 20}{2}=21\times 10=$ $20=210$.

And the sum of n *terms of* a, $a+x$, $a+2x$, $a+3x$, &c. *to* $a+mx$, *is* $=\frac{a+a+m\times n}{2}=a+\frac{mn}{2}$.

GEOMETRICAL PROPORTION.

Geometrical proportion is that relation of two quantities, of the ſame kind, which ariſes from conſidering what part the one is of the other, or how often it is contained in it.

In four proportional quantities, the firſt and third are called the *antecedents*, and the ſecond and fourth the *conſequents*.

Ratio is the quotient which ariſes from dividing the antecedent by the conſequent, or the conſequent by the antecedent.

Four quantities are ſaid to be proportional, when the firſt is the ſame part or multiple of the ſecond, as the third is of the fourth.

Thus, 2, 8, 3, 12, *and* a, ar, b, br, *are geometrical proportionals*.

Direct proportion is when the ſame relation ſubſiſts between the firſt term and the ſecond, as between the third and the fourth.

Thus, 3, 6, 5, 10, *and* x, ax, y, ay, *are in direct proportion*.

Reciprocal, or inverſe proportion, is when one quantity increaſes in the ſame proportion that another diminiſhes.

Thus, 2, 6, 9, 3, *and* a; ar, br, b; *are in inverſe proportion*.

A ſeries of quantities are ſaid to be in geometrical progreſſion, when the firſt has the ſame ratio to the ſecond, as the ſecond to the third, and as the third to the fourth, &c.

Thus, 2, 4, 8, 16, 32, 64, &c. *and* a, ar, ar^2, ar^3, ar^4, ar^5, *&c. are series in geometrical progression.*

The most necessary part of geometrical proportion is contained in the following theorems.

I. If four quantities be in geometrical proportion, the product of the two means will be equal to that of the two extremes.

Thus, if 2, 4, 6, 12, *and* a, ar, b, br, *be geometrically proportional, then will* $2\times12=4\times6$, *and* $a\times br=b\times ar$.

II. If four quantities be in geometrical proportion, the rectangle of the means divided by either of the extremes will give the other extreme.

Thus, if 3, 9, 5, 15, *and* x, ax, y, ay, *are geometricalproportionals, then will* $\frac{9\times5}{3}=15$, *and* $\frac{ax\times y}{ay}=x$.

III. The sum of any series of quantities in geometrical progression, is found by multiplying the last term by the ratio, and dividing the difference of this product and the first term by the ratio less one.

Thus, the sum of 2, 4, 8, 16, 32, 64, 128, 256, 512, *is* $\frac{512\times2-2}{2-1}=1024-2=1022$.

And the sum of n terms of a, ar, ar^2, ar^3, ar^4, *&c, to* ar^{n-1}, *is* $\frac{ar^{n-1}\times r-a}{r-1}=\frac{ar^n-a}{r-1}$.

IV. If four quantities, a, b, c, d, or 2, 6, 5, 15, are proportional, then will any of the following forms of those quantities be also proportional.

1. *directly* $a:b::c:d$ or $2:6::5:15$.
2. *inversely* $b:a::d:c$ or $6:2::15:5$.
3. *alternately* $a:c::b:d$ *or* $2:5+6:15$.
4. *compoundedly* $a:a+b::c:c+d$ *or* $2:8::5:20$.
5. *dividedly* $a:b-a::c:d-c$ *or* $2:4::5:10$.
6. *mixtly* $b+a:b-a::d+c:d-c$ *or* $8:4::20:10$.
7. *by multiplication* $ra:rb::c:d$ *or* $2\times3(6):6\times3(18)::5:15$.

8. *by division* $\frac{a}{r} : \frac{b}{r} :: c : d$ *or* $\frac{2}{2}(1) : \frac{6}{2}(3) ::$ $5 : 15$.

SIMPLE EQUATIONS.

An equation is, when two equal quantities, differently expressed, are compared together, by means of the sign $=$ placed between them.

Thus $12-5=7$ is an equation, expressing the equality of the quantities $12-5$ and 7.

A simple equation is that which contains only one unknown quantity, without including its power.

Thus $x-a+b=c$ is a simple equation, containing only the unknown quantity x.

Reduction of equations is the method of finding the value of the unknown quantity; which is shewn in the following rules.

RULE I.

Any quantity may be transposed from one side of the equation to the other, by changing its sign.

Thus, if $x+3=7$, *then will* $x=7-3=4$.
And, if $x-4+6=8$, *then will* $x=8+4-6=6$.
Also, if $x-a+b=c-d$, *then will* $x=c-d+a-b$.
And, in like manner, if $4x-8=3x+20$, *then will* $4x-3x=20+8$, *or* $x=28$.

RULE II.

If the unknown term be multiplied by any quantity, it may be taken away by dividing all the other terms of the equation by it.

Thus, if $ax=ab-a$, *then will* $x=b-1$.

And, if $2x+4=16$, *then will* $x+2=8$, *and* $x=8-2=6$.

In like manner, if $ax+2ba=3c^2$, *then will* $x+2b=\frac{3c^2}{a}$, *and* $x=\frac{3c^2}{a}-2b$.

RULE III.

If the unknown term be divided by any quantity, it may be taken away by multiplying all the other terms of the equation by it.

Thus, if $\frac{x}{2}=5+3$, *then will* $x=10+6=19$.

And, if $\frac{x}{a}=b+c-d$, *then will* $x=ab+ac-ad$.

In like manner, if $\frac{2x}{3}-2=6+4$, *then will* $2x-6=18+12$, *and* $2x=18+12+6=36$, *or* $x=\frac{36}{2}=18$.

RULE IV.

The unknown quantity in any equation may be made free from surds, by transposing the rest of the terms according to the rule, and then involving each side to such a power as is denoted by the index of the said surd.

Thus, if $\sqrt{x}-2=6$, *then will* $\sqrt{x}=6+2=8$, *and* $x=8^2=64$.

And, if $\sqrt{4x+16}=12$, *then will* $4x+16=144$, *and* $4x=144-16=128$, *or* $x=\frac{128}{4}=32$.

In like manner, if $\sqrt[3]{2x+3}+4=8$, *then will* $\sqrt[3]{2x+3}=8-4=4$, *and* $2x+3=4^3=64$, *and* $2x=64-3=61$, *or* $x=\frac{61}{2}=30\frac{1}{2}$.

RULE V.

If that side of the equation which contains the unknown quantity be a complete power, it may be reduced by extracting the root of the said power from both sides of the equation.

Thus, if $x^2+6x+9=25$, *then will* $x+3=\sqrt{25}=5$, *or* $x=5-3=2$.

And, if $3x^2-9=21+3$, *then will* $3x^2=21+3+9=33$, *and* $x^2=\frac{33}{3}=11$, *or* $x=\sqrt{11}$.

In like manner, if $\frac{2x^2}{3}+10=20$, *then will* $2x^2+30=60$, *and* $x^2+15=30$, *or* $x^2=30-15=15$, *or* $x=\sqrt{15}$.

RULE VI.

Any analogy, or proportion, may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.

Thus, if $3x:16::5:10$, *then will* $3x\times10=16\times5$, *and* $30x=80$, *or* $x=\frac{80}{30}=2\frac{2}{3}$.

And, if $\frac{2x}{3}:a::b:c$, *then will* $\frac{2cx}{3}=ab$, *and* $2cx=3ab$, *or* $x=\frac{3ab}{2c}$.

In like manner, if $12-x:\frac{x}{2}::4:1$, *then will* $12-x=\frac{4x}{2}=2x$, *and* $2x+x=12$, *or* $x=\frac{12}{3}=4$.

RULE VII.

If any quantity be found on both sides of the equation with the same sign, it may be taken away from them both; and if every term in an equation be multiplied or divided by the same quantity, it may be struck out of them all.

Thus, if $4x+a=b+a$, *then will* $4x=b$, *and* $x=\frac{b}{4}$.

And, if $3ax+5ab=8ac$, *then will* $3x+5b=8c$, *and* $x=\frac{8c-5b}{3}$.

In like manner, if $\frac{2x}{3}-\frac{8}{3}=\frac{16}{3}-\frac{8}{3}$, *then will* $2x=16$, *and* $x=8$.

MISCELLANEOUS EXAMPLES.

1. Given $5x-15=2x+6$ to find the value of x.

   First $5x-2x=6+15$
   then $3x=21$
   and $x=\frac{21}{3}=7$

2. Given $40-6x-16=120-14x$ to find x.

   First $14x-6x=120-40+16$
   then $8x=96$
   and, therefore $x=\frac{96}{8}=12$.

3. Let $5ax-3b=2dx+c$ be given, to find x.

First, $5ax-2dx=c+3b$

or $\overline{5a-2d}\times x=c+3b$

and therefore $x=\dfrac{c+3b}{5a-2d}$.

4. Let $3x^2-10x=8x+x^2$ be given to find x.

First, $3x-10=8+x$

and then $3x-x=8+10$

therefore $2x=18$, *and* $x=\dfrac{18}{2}=9$.

5. Given $6ax^3-12abx^2=3ax^3+6ax^2$, to find x.

First, dividing the whole by $3ax^2$, *we shall have*

$2x-4b=x+2$

and then $2x-x=2+4b$

whence $x=2+4b$.

6. Let $\dfrac{x}{2}-\dfrac{x}{3}+\dfrac{x}{4}=10$, be given to find x.

First, $x-\dfrac{2x}{3}+\dfrac{2x}{4}=20$

and then $3x-2x+\dfrac{6x}{4}=60$

and $12x-8x+6x=240$

therefore $10x=240$

and $x=\dfrac{240}{10}=24$.

7. Given $\dfrac{x-3}{2}+\dfrac{x}{3}=20-\dfrac{x+19}{2}$ to find x.

First, $x-3+\dfrac{2x}{3}=40-x-19$

and then $3x-9+2x=120-3x-57$

and therefore $3x+2x+3x=120-57+9$

that is $8x=72$, *or* $x=\dfrac{72}{8}=9$.

8. Let $\sqrt{\frac{2}{3}x+5}=7$, be given to find x.

First, $\sqrt{\frac{2}{3}x}=7-5=2$

and then $\frac{2}{3}x=2^2=4$

and $2x=12$, *or* $x=\frac{12}{2}=6$.

9. Let $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$ be given to find x.

First, $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$

and then $x\sqrt{a^2+x^2}=a^2-x^2$

and $x^2\times\overline{a^2+x^2}=\overline{a^2-x^2}|^2=a^4-2a^2x^2+x^4$

or $a^2x^2+x^4=a^4-2a^2x^2+x^4$

whence $a^2x^2+2a^2x^2=a^4$, *or* $3a^2x^2=a^4$

and consequently $x^2=\frac{a^4}{3a^2}$, *and* $x=\sqrt{\frac{a^4}{3a^2}}=a\sqrt{\frac{1}{3}}$

EXAMPLES FOR PRACTICE.

1. Given $x+18=3x-5$ to find x. *Ans.* $x=11\frac{1}{2}$.
2. Given $3y-a+b=cd$ to find y.

Ans. $y=\frac{cd+a-b}{3}$.

3. Given $6-2x+10=20+3x-2$ to find x.

Ans. $x=2$.

4. Given $3ax+\frac{a}{2}-3=bx-a$ to find x.

6a *Ans.* $x=\frac{6-3a}{6a-2b}$.

5. Given $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d$ to find x.

Ans. $x=\frac{abcd}{bc+ac+ab}$.

6. Given $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{1}{2}$ to find x.

Ans. $x=\frac{6}{7}$.

7. Given $\sqrt{12+x}=2+\sqrt{x}$ to find x.

Ans. $x=4$.

8. Given $\sqrt{a^2+x^2}=\overline{b^4+x^4}|^{\frac{1}{4}}$ to find x.

Ans. $x=\sqrt{\frac{b^4-a^4}{2a^2}}$.

9. Given $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}$ to find x.

Ans. $x=\frac{b^2}{4a}-a$.

PROBLEM I.

To exterminate two unknown quantities, or to reduce the two simple equations containing them to a single one.

RULE I.

1. Obſerve which of the unknown quantities is the leaſt involved, and find its value in each of the equations, by the methods already explained.

2. Let the two values thus found be made equal to each other, and there will ariſe a new equation with only one unknown quantity in it, whoſe value may be found as before.

EXAMPLES:

1. Given $\left\{\begin{array}{l}2x+3y=23\\5x-2y=10\end{array}\right\}$ to find x and y.

From the first equation $x=\frac{23-3y}{2}$,

and from the second $x=\frac{10+2y}{5}$,

and consequently $\frac{23-3y}{2}=\frac{10+2y}{5}$,

or $115-15y=20+4y$,

or $19y=115-20=95$,

and $y=\frac{95}{19}=5$,

whence $x=\frac{23-15}{2}=4$.

2. Given $\left\{\begin{array}{l}x+y=a\\x-y=b\end{array}\right\}$ to find x and y.

From the first equation $x=a-y$,
and from the second $x=b+y$,
Therefore $a-y=b+y$, *or* $2y=a-b$,

and consequently $y=\frac{a-b}{2}$,

and $x=a-y=a-\frac{a-b}{2}=\frac{a+b}{2}$.

3. Given $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}=7\\\frac{x}{3}+\frac{y}{2}=8\end{array}\right\}$ to find x and y.

From the first equation $x=14-\frac{2y}{3}$,

and from the second $x=24-\frac{3y}{2}$,

Therefore $14-\frac{2y}{3}=24-\frac{3y}{2}$,

and $42-2y=72-\frac{9y}{2}$,

or $84-4y=144-9y$;

whence $5y=144-84=60$,

and $y=\frac{60}{5}=12$,

and $x=14-\frac{2y}{3}=14-\frac{24}{3}=6$.

4. Given $4x+y=34$, and $4y+x=16$, to find x and y. *Anſ.* $x=8$, *and* $y=2$.

5. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{2x}{4}+\frac{2y}{5}=\frac{61}{120}$, to find x and y. *Anſ.* $x=\frac{1}{2}$ *and* $y=\frac{1}{3}$.

6. Given $x+y=s$, and $x^2-y^2=d$, to find x and y. *Anſ.* $x=\frac{s^2+d}{2s}$, *and* $y=\frac{s^2-d}{2s}$.

RULE II.

1. Conſider which of the unknown quantities you would firſt exterminate, and let its value be found in that equation where it is the leaſt involved.

2. Subſtitute the value, thus found, for its equal in the other equation, and there will ariſe a new equation, with only one unknown quantity, whoſe value may be found as before.

EXAMPLES:

1. Given $\left\{\begin{array}{l} x+2y=17 \\ 3x-y=2 \end{array}\right\}$ to find x and y.

From the first equation $x=17-2y$,
And this value substituted for x *in the second, gives*
$\overline{17-2y}\times 3-y=2$,
or $51-6y-y=2$, *or* $51-7y=2$;
that is $7y=51-2=49$;
whence $y=\frac{49}{7}=7$, *and* $x=17-2y=17-14=2$.

2. Given $\left\{\begin{array}{l} x+y=13 \\ x-y=3 \end{array}\right\}$ to find x and y.

From the first equation $x=13-y$,
And this value being substituted for x *in the 2d,*
gives $13-y-y=3$, *or* $13-2y=3$,
that is $2y=13-3=10$,
or $y=\frac{10}{2}=5$, *and* $x=13-y=13-5=8$.

3. Given $\left\{\begin{array}{l} a:b::x:y \\ x^2+y^2=c \end{array}\right\}$ to find x and y.

The first analogy turned into an equation
is $bx=ay$, *or* $x=\frac{ay}{b}$,
And this value of x *substituted in the 2d,*
gives $\left.\overline{\frac{ay}{b}}\right|^2+y^2=c$, *or* $\frac{a^2y^2}{b^2}+y^2=c$,
or $a^2y^2+b^2y^2=cb^2$, *or* $y^2=\frac{cb^2}{a^2+b^2}$,
and therefore $y=\left.\overline{\frac{cb^2}{a^2+b^2}}\right|^{\frac{1}{2}}$, *and* $x=\left.\overline{\frac{ca^2}{a^2+b^2}}\right|^{\frac{1}{2}}$.

4. Given $\frac{x}{7}+7y=99$, *and* $\frac{y}{7}+7x=51$, to find x and y. *Ans.* $x=7$ *and* $y=14$.

5. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, to find x and y. *Ans.* $x=60$ *and* $y=40$.

6. Given $a:b::x:y$, and $x^3-y^3=d$, to find x and y.

Ans. $\left.\frac{da^3}{a^3-b^3}\right|^{\frac{1}{3}}=x$, *and* $\left.\frac{db^3}{a^3-b^3}\right|^{\frac{1}{3}}=y$.

RULE III.

Let the given equations be multiplied or divided by ſuch numbers or quantities as will make the term which contains one of the unknown quantities to be the ſame in both equations; and then by adding or ſubtracting the equations, according as is required, there will ariſe a new equation with only one unknown quantity, as before.

EXAMPLES.

1. Given $\left\{\begin{array}{c}3x+5y=40\\ x+2y=14\end{array}\right\}$ to find x and y.

Firſt, multiply the 2d equation by 3,
and we ſhall have $3x+6y=42$,
Then, from this laſt equation ſubtract the firſt,
and it will give $6y-5y=42-40$, *or* $y=2$,
and therefore $x=14-2y=14-4=10$.

2. Given $\left\{\begin{array}{l} 5x-3y=9 \\ 2x+5y=16 \end{array}\right\}$ to find x and y.

Let the 1st equation be multiplied by 2, and the 2d by 5, and we shall have $10x-6y=18$
$10x+25y=80$
and if the former of these be subtracted from the latter, it will give $31y=62$, *or* $y=\frac{62}{31}=2$,

and consequently $x=\frac{9+3y}{5}$, *by the first equation,*

or $x=\frac{9+6}{5}=\frac{15}{5}=3$.

Another method.

Multiply the first equation by 5, and the second by 3, and we shall have $25x-15y=45$
$6x+15y=48$
Now, let these two equations be added together,

and it will give $31x=93$, *or* $x=\frac{93}{31}=3$,

and consequently $y=\frac{16-2x}{5}$, *by the 2d equation,*

or $y=\frac{16-6}{5}=\frac{10}{5}=2$ *as before.*

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x+2}{3}+8y=31$, and $\frac{y+5}{4}+10x=192$, to find x and y. *Ans.* $x=19$ *and* $y=3$.

2. Given $\frac{2x-y}{2}+14=18$, and $\frac{2y+x}{3}+16=19$ to find x and y. *Ans.* $x=5$ *and* $y=2$.

3. Given $\frac{2x+3y}{6}+\frac{x}{3}=8$, and $\frac{7y-3x}{2}-y=11$, to find x and y. *Anſ.* $x=6$ *and* $y=8$.

4. Given $ax+by=c$, and $dx+ey=f$, to find x and y. *Anſ.* $x=\frac{ce-bf}{ae-db}$ *and* $y=\frac{af-dc}{ae-db}$.

PROBLEM II.

To exterminate three unknown quantities, or to reduce the three ſimple equations, containing them, to a ſingle one.

RULE.

1. Let x, y, and z, be the three unknown quantities to be exterminated.

2. Find the value of x from each of the three given equations.

3. Compare the firſt value of x with the ſecond, and an equation will ariſe involving only y and z.

4. In like manner, compare the firſt value of x with the third, and another equation will ariſe involving only y and z.

5. Find the values of y and z from theſe two equations, according to the former rules, and x, y, and z will be exterminated as required.

Note, Any number of unknown quantities may be exterminated in nearly the ſame manner, but there are often much ſhorter methods for performing the operation, which will be beſt learnt from practice.

EXAMPLES:

1. Given $\left\{\begin{array}{l} x+y+z=29 \\ x+2y+3z=62 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10 \end{array}\right\}$ to find x, y, and z.

From the first equation $x=29-y-z$.

From the second $x=62-2y-3z$.

From the third $x=20-\frac{2y}{3}-\frac{z}{2}$.

whence $29-y-z=62-2y-3z$,

and $29-y-z=20-\frac{2y}{3}-\frac{z}{2}$,

but, from the first of these equations, $y=62-29-2z=33-2z$,

and from the second $y=27-\frac{3z}{2}$,

therefore $33-2z=27-\frac{3z}{2}$, *or* $z=12$,

and $y=62-29-2z=62-29-24=9$.

and $x=29-y-z=29-12-9=8$.

2. Given $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38 \end{array}\right\}$ to find x, y, and z.

First, the given equations, cleared of fractions, become

$$12x+8y+6z=1488$$
$$20x+15y+12z=2820$$
$$30x+24y+20z=4560.$$

Then, if the second of these equations be subtracted from double the first, and three times the third from five times the second, we shall have

$$4x+y=156$$
$$10x+3y=420$$

And again, if the second of these be subtracted from three times the first, it will give

$$12x-10x=468-420, \text{ or } x=\frac{48}{2}=24$$

Therefore $y=156-4x=60$, *and* $z=\frac{1488-8y-12x}{6}$ $=120$.

3. Given $x+y+z=31$, $x+y-z=25$, and $x-y-z=9$, to find x, y, and z.

Ans. $x=20$, $y=8$, *and* $z=3$.

4. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find x, y, and z.

5. Given $\left\{\begin{array}{l} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{array}\right\}$ to find x, y, and z.

A COLLECTION OF QUESTIONS

PRODUCING SIMPLE EQUATIONS.

1. To find two numbers, such that their sum shall be 40, and their difference 16.

Let x *denote the least of the two numbers required,*
then will $x+16=$ *to the greater,*
and $x+x+16=40$ *by the question,*
that is $2x=40-16=24$,

or $x=\frac{24}{2}=12=$ *least number,*

and $x+16=12+16=28=$ *greater number required.*

2. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let $x=$ *number required,*

then will its $\frac{1}{3}$ *part be* $\frac{x}{3}$, *and its* $\frac{1}{4}$ *part* $\frac{x}{4}$;

and therefore $\frac{x}{3}-\frac{x}{4}=16$ *by the question,*

that is $x-\frac{3x}{4}=48$, *or* $4x-3x=192$;

whence $x=192$ *the number required.*

3. Divide 1000l. between A, B, and C, so that A shall have 72l. more than B, and C 100l. more than A.

Let $x=$ *B's share of the given sum,*
then will $x+72=$ *A's share,*
and $x+172=$*B's share,*
And the sum of all their shares $x+x+72+x+172$,
or $3x+244=1000$ *by the question,*
that is $3x=1000-244=756$,
or $x=\frac{756}{3}=252$l.$=$ *B's share,*
and $x+72=252+72=324$l.$=$*A's share,*
and $x+172=252+172=424$l.$=$*C's share.*

252l.
324l.
424l.

1000l. *the proof.*

5. A prize of 1000l. is to be divided between two persons, whose shares therein are in the proportion of 7 to 9: required the share of each.

Let $x=$ *first person's share,*
then will $1000-x=$ *second person's share,*
and $x:1000-x::7:9$, *by the question,*
that is $9x=\overline{1000-x}\times 7=7000-7x$,
or $16x=7000$,
whence $x=\frac{7000}{16}=437$l. 10s. $=$ *1st share,*
and $1000-x=1000-437$l. 10s. $=562$l. 10s. *2d share.*

6. The paving of a square at 2s. a yard, cost as much as the inclosing it at 5s. a yard: required the side of the square.

Let $x=$ *side of the square sought,*
then $4x=$ *yards of inclosure,*
and $x^2=$ *yards of pavement;*
whence $4x\times 5=20x=$ *price of inclosing,*
and $x^2\times 2=2x^2=$ *price of paving.*
But $2x^2=20x$ *by the question,*
therefore $x^2=10x$, *and* $x=10=$ *length of the side required.*

7. A labourer engaged to serve for 40 days upon these conditions, that for every day he worked he should receive 20 d. but for every day that he played, or was absent, he was to forfeit 8 d. now at the end of the time he had to receive 1 l. 11 s. 8 d. the question is, to find how many days he worked, and how many he was idle.

Let x *be the number of days he worked,*
then will $40-x$ *be the number of days he was idle,*
also $x\times 20=20x=$ *sum earned,*
and $\overline{40-x}\times 8=320-8x=$ *sum forfeited,*
whence $20x-\overline{320-8x}=380d.$ ($=1\,l.\ 11\,s.\ 8d.$) *by the question, that is* $20x-320+8x=380$
or $28x=380+320=700$,

and $x=\frac{700}{28}=25=$ *number of days he worked,*

and $40-x=40-25=15=$ *number of days he was idle.*

8. Out of a cask of wine, which had leaked away $\frac{1}{3}$, 21 gallons were drawn; and then, being gauged, it appeared to be half full: How much did it hold?

Let it be supposed to have held x gallons,

Then it would have leaked $\frac{x}{3}$ *gallons,*

and consequently there had been taken away $21+\frac{x}{3}$ *gallons.*

But $21+\frac{x}{3}=\frac{x}{2}$ *by the question,*

That is $63+x=\frac{3x}{2}$,

or $126+2x=3x$

hence $3x-2x=126$

or $x=126=$ *number of gallons required.*

9. What fraction is that, to the numerator of which if 1 be added, the value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction be represented by $\frac{x}{y}$,

Then will $\frac{x+1}{y}=\frac{1}{3}$,

And $\frac{x}{y+1}=\frac{1}{4}$,

or $3x+3=y$,

and $4x=y+1$,

hence $4x-3x-3=y+1-y$
that is $x-3=1$
or $x=4$, *and* $y=3x+3$
$=12+3=15$

So that $\frac{4}{15}=$ *fraction required.*

10. A market-woman bought in a certain number of eggs, at 2 a penny, and as many at three a penny, and ſold them all out again at the rate of 5 for 2 pence, and, by ſo doing, loſt 4 d. what number of eggs had ſhe?

Let $x=$ *number of eggs of each ſort.*

Then will $\frac{x}{2}=$ *price of the firſt ſort,*

and $\frac{x}{3}=$ *price of the ſecond ſort.*

But $5:2::2x$ (*the whole number of eggs*) $:\frac{4x}{5}$;

therefore $\frac{4x}{5}$ *price of both ſorts together, at five for 2d.*

And $\frac{x}{2}+\frac{x}{3}-\frac{4x}{5}=4$ *by the queſtion;*

that is $x+\frac{2x}{3}-\frac{8x}{5}=8$;

or $3x+2x-\frac{24x}{5}=24$;

or $15x+10x-24x=120$

Whence $x=120=$ *number of eggs of each ſort required.*

11. If A can do a piece of work alone in ten days, and B in thirteen; ſet them both about it together, in what time will it be finiſhed?

Let the time fought be denoted by x,

Then 10 *days* : 1 *work* :: *x days* : $\frac{x}{10}$,

and 13 *days* : 1 *work* :: *x days* : $\frac{x}{13}$,

hence $\frac{x}{10}$ = *part done by A in x days*;

and $\frac{x}{13}$ = *part done by B in x days.*

Confequently $\frac{x}{10}+\frac{x}{13}=1$;

That is $\frac{13x}{10}+x=13$, *or* $13x+10x=130$;

And therefore $23x=130$, *or* $x=\frac{130}{23}=5\frac{15}{23}$ *days, the time required.*

12. If one agent A, alone, can produce an effect e, in the time a, and another agent B, alone, in the time b; in what time will they both together produce the fame effect?

Let the time fought be denoted by x.

Then $a:e::x:\frac{ex}{a}$ = *part of the effect produced by A,*

and $b:e::x:\frac{ex}{b}$ = *part of the effect produced by B,*

whence $\frac{ex}{a}+\frac{ex}{b}=e$ *by the queftion*;

or $\frac{x}{a}+\frac{x}{b}=1$;

that is $x+\frac{ax}{b}=a$;

or $bx+ax=ba$;

and confequently $x=\frac{ba}{b+a}$ = *time required.*

QUESTIONS FOR PRACTICE.

1. What two numbers are thoſe whoſe difference is 7, and ſum 33. *Anſ.* 13 *and* 20.

2. To divide the number 75 into two ſuch parts, that three times the greater may exceed ſeven times the leſſer by 15. *Anſ.* 54 *and* 21.

3. In a mixture of wine and cyder, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons was cyder: how many gallons were there of each?

Anſ. 85 *of wine, and* 35 *of cyder.*

4. A bill of 120l. was paid in guineas and moidores, and the number of pieces of both ſorts that were uſed was juſt 100; how many were there of each? *Anſ.* 50 *of each.*

5. Two travellers ſet out at the ſame time from London and York, whoſe diſtance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet? *Anſ. in* 10 *days.*

6. At a certain election 375 perſons voted, and the candidate choſen had a majority of 91; how many voted for each?

Anſ. 233 *for one, and* 142 *for the other.*

7. There is a fiſh whoſe tail weighs 9 ℔. his head weighs much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fiſh?

Anſ. 72 ℔.

8. What number is that from which if 5 be ſubtracted, $\frac{2}{3}$ of the remainder will be 40. *Anſ.* 65.

9. A poſt is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length?

Anſ. 24 *feet.*

10. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I found 66 guineas left in my bag; what was in it at firſt? *Anſ.* 120 *guineas.*

11. A's age is double of B's, and B's is triple of C's, and the ſum of all their ages is 140; what is the age of each?

Anſ. A's=84, *B's*=42, *and C's*=14.

12. Two perſons, A and B, lay out equal ſums of money in trade; A gains 126 l. and B loſes 87 l. and A's money is now double of B's; what did each lay out? *Anſ.* 300 *l.*

13. A perſon bought a chaiſe, horſe and harneſs, for 60 l. the horſe came to twice the price of the harneſs, and the chaiſe to twice the price of the horſe and the harneſs; what did he give for each? *Anſ.* 13 *l.* 6 *s.* 8 *d. for the horſe,* 6 *l.* 13 *s.* 4 *d. for the harneſs, and* 40 *l. for the chaiſe.*

14. Two perſons, A and B, have both the ſame income; A ſaves $\frac{1}{5}$ of his yearly, but B, by ſpending 50 l. *per annum* more than A, at the end of 4 years finds himſelf 100 l. in debt; what is their income? *Anſ.* 125 *l.*

15. A gentleman has two horſes, and a ſaddle worth 50 l. now if the ſaddle be put on the back of the firſt horſe, it will make his value double that of the ſecond; but if it be put on the back of the ſecond, it will make his value triple that of the firſt; what is the value of each horſe?

Anſ. One 30 *l. and the other* 40 *l.*

16. To divide the number 36 into three ſuch parts, that $\frac{1}{2}$ of the firſt, $\frac{1}{3}$ of the ſecond, and $\frac{1}{4}$ of the third, may be all equal to each other?

Anſ. The parts are 8, 12, *and* 16.

17. A footman agreed to ſerve his maſter for 8 l. a year and a livery, but was turned away at the end of 7 months, and received only 2 l. 13 s. 4 d. and his livery; what was its value? *Anſ.* 4 *l.* 16 *s.*

18. A gentleman was desirous of giving 3 d. a-piece to some poor beggars, but found that he had not money enough in his pocket by 8 d. he therefore gave them each 2 d. and had then 3 d. remaining; required the number of beggars?

Ans. 11.

19. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's three; but 2 of the greyhound's leaps are as much as three of the hare's; how many leaps must the greyhound take to catch the hare? *Ans.* 300.

20. A person in play, lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which, he lost $\frac{1}{3}$ of what he then had, and then won 2 shillings; lastly he lost $\frac{1}{7}$ of what he then had, and this done, found he had but 12 s. remaining; what had he at first.

Ans. 20 *s.*

21. To divide the number 90 into 4 such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2; the sum, difference, product, and quotient, shall be all equal to each other.

Ans. The parts are 18, 22, 10, and 40 *respectively.*

22. The hour and minute hand of a clock are exactly together at 12 o'clock, when are they next together? *Ans.* 1 *ho.* $5\frac{5}{11}$ *min.*

23. There is an island 73 miles in circumference, and three footmen all start together to travel the same way about it: A goes 5 miles a-day, B 8, and C 10; when will they all come together again?

Ans. 73 *days.*

24. If A can do a piece of work alone in 10 days, and A and B together in 7 days; in what time can B do it alone? *Ans.* $23\frac{1}{3}$ *days.*

25. If three agents, A, B, and C, can produce the effects a, b, c, in the times e, f, g, respectively; in what time would they jointly produce the effect d?

Ans. $\frac{\overline{e+f+g}\times d}{a+b+c}$ *time.*

26. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days: How many days will it take each person to perform the same work alone?

Ans. *A* $14\frac{34}{49}$ *days*, *B* $17\frac{23}{41}$, *and C* $23\frac{7}{31}$.

QUADRATIC EQUATIONS.

A simple quadratic equation is that which involves the square of the unknown quantity only.

An adfected quadratic equation is that which involves the square of the unknown quantity, together with the product that arises from multiplying it by some known quantity.

Thus $ax^2=b$, *is a simple quadratic equation,*

And $ax^2+bx=c$, *is an adfected quadratic equation.*

The rule for a simple quadratic equation has been given already.

All adfected quadratic equations fall under the three following forms.

1. $x^2+ax=b$
2. $x^2-ax=b$
3. $x^2-ax=-b$.

The rule for finding the value of x, in each of these equations, is as follows:

RULE.*

1. Transpose all the terms that involve the unknown quantity to one side of the equation, and the known terms to the other side, and let them be ranged according to their dimensions.

2. When the square of the unknown quantity has any co-efficient prefixed to it, let all the rest of the terms be divided by that co-efficient.

3. Add the square of half the co-efficient of the second term to both sides of the equation, and that side which involves the unknown quantity will then be a complete square.

4. Extract the square root from both sides of the equation, and the value of the unknown quantity will be determined, as required.

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* The square root of any quantity may be either $+$ or $-$, and therefore all quadratic equations admit of two solutions. Thus the square root of $+n^2$ is $+n$ or $-n$; for $+n\times+n$ or $-n\times-n$ are each equal to $+n^2$. So, in the first form, where $x+\frac{a}{2}$ is found $=\sqrt{b+\frac{a^2}{4}}$ the root may be either $+\sqrt{b+\frac{a^2}{4}}$ or $-\sqrt{b+\frac{a^2}{4}}$, since either of them being multiplied by itself will produce $\sqrt{b+\frac{a^2}{4}}$. # And this ambiguity is expressed by writing the uncertain sign $\pm$ before $\sqrt{b+\frac{a^2}{4}}$; thus $x=\pm\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$.

should be $b+\frac{aa}{4}$, having no radical sign ($\sqrt{}$).

Note, 1. The square root of one side of the equation is always equal to the unknown quantity, with half the co-efficient of the second term subjoined to it.

2. All equations wherein there are two terms involving the unknown quantity, and the index of the one is just double that of the other, are solved like quadratics, by completing the square.

Thus, $x^4+ax^2=b$, *or* $x^n+ax^{\frac{n}{2}}=b$, *are the same as quadratics, and the value of the unknown quantity may be determined accordingly.*

In the first form, where $x=\pm\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$, the first value of x, viz. $x=+\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ is always affirmative; for since $\frac{a^2}{4}+b$ is greater than $\frac{a^2}{4}$, the greatest square must necessarily have the greatest square root; $\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ will, therefore, always be greater than $\sqrt{\frac{a^2}{4}}$, or its equal $\frac{a}{2}$; and consequently $+\sqrt{b+\frac{a^2}{2}}-\frac{a}{2}$ will always be affirmative.

The second value, viz. $x=-\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ will always be negative, because it is composed of two negative terms. Therefore when $x^2+ax=b$, we shall have $x=+\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ for the affirmative value of x, and $x=-\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ for the negative value of x.

EXAMPLES:

1. Given $x^2+4x=140$ to find x.

First, $x^2+4x+4=140+4=144$, *by completing the square*;

then $\sqrt{x^2+4x+4}=\sqrt{144}$ *by extracting the root*;

or $x+2=12$,

and therefore $x=12-2=10$.

2. Given $x^2-6x+8=80$, to find x.

First, $x^2-6x=80-8=72$ *by transposition*;

Then $x^2-6x+9=72+9=81$ *by completing the square*;

And $x-3=\sqrt{81}=9$ *by extracting the root*;

Therefore $x=9+3=12$.

In the second form, where $x=\pm\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ the first value, viz. $x=+\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ is always affirmative, since it is composed of two affirmative terms. The second value, viz. $x=-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ will always be negative; for since $b+\frac{a^2}{4}$ is greater than $\frac{a^2}{4}$, the square root of $b+\frac{a^2}{4}$ $\left(\sqrt{b+\frac{a^2}{4}}\right)$ will be greater than $\sqrt{\frac{a^2}{4}}$, or its equal $\frac{a}{2}$; and consequently $-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ is always a negative quantity. Therefore when $x^2-ax=b$, we shall have $x=+\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ for the affirmative value of x, and $x=-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ for the negative value of x.

3. Given $2x^2+8x-20=70$, to find x.

First, $2x^2+8x=70+20=90$ *by transposition.*
Then $x^2+4x=45$ *by dividing by* 2,
And $x^2+4x+4=49$ *by completing the square;*
Whence $x+2=\sqrt{49}=7$ *by extracting the root.*
And consequently $x=7-2=5$.

4. Given $3x^2-3x+6=5\frac{1}{3}$, to find x.

Here, $x^2-x+2=1\frac{7}{9}$ *by dividing by* 3,
And $x^2-x=1\frac{7}{9}-2$ *by transposition;*
Also $x^2-x+\frac{1}{4}=1\frac{7}{9}-2+\frac{1}{4}=\frac{1}{36}$ *by completing the squ.*
And $x-\frac{1}{2}=\sqrt{\frac{1}{36}}=\frac{1}{6}$ *by evolution;*
Therefore $x=\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$.

In the third form, where $x=\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$, both the values of x will be positive, supposing $\frac{a^2}{4}$ is greater than b. For the first value, viz. $x=+\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ is evidently affirmative, being composed of two affirmative terms. The second value, viz. $x-\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ is also affirmative; for since $\frac{a^2}{4}$ is greater than $\frac{a^2}{4}-b$, therefore $\sqrt{\frac{a^2}{4}}$ or $\frac{a}{2}$ is greater than $\sqrt{\frac{a^2}{4}-b}$, and consequently $-\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ will always be an affirmative quantity. Therefore, when $x^2-ax=-b$, we shall have $x=+\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ for the affirmative value of x, and $-\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ for the negative value of x.

5. Given $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$, to find x.

Here $\frac{x^2}{2}-\frac{x}{3}=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$ *by transposition*,

And $x^2-\frac{2x}{3}=44\frac{1}{3}$ *by dividing by* $\frac{1}{2}$;

Whence $x^2-\frac{2x}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=44\frac{4}{9}$ *by completing the square*,

And $x-\frac{1}{3}=\sqrt{44\frac{4}{9}}=6\frac{2}{3}$,

Therefore $x=6\frac{2}{3}+\frac{1}{3}=7$.

6. Given $ax^2+bx=c$, to find x.

First, $x^2+\frac{b}{a}x=\frac{c}{a}$ *by division*;

Then $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}$ *by completing the square*;

But in this third form, if b be greater than $\frac{a^2}{4}$ the solution of the proposed question will be impossible. For since the square of any quantity (whether that quantity be affirmative or negative) is always affirmative, the square root of a negative quantity is impossible, and cannot be assigned. But if b be greater than $\frac{a^2}{4}$, then $\frac{a^2}{4}-b$ is a negative quantity; and consequently $\sqrt{\frac{a^2}{4}-b}$ is impossible, or only imaginary, when $\frac{a^2}{4}$ is less than b; and therefore in that case $x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-b}$ is also impossible or imaginary.

And $x+\frac{b}{2a}=\sqrt{\frac{c}{a}+\frac{b^2}{4a^2}}=\sqrt{\frac{4ac+b^2}{4a^2}}$ *by evolution;*

Therefore $x=\pm\sqrt{\frac{4ac+b^2}{4a^2}}-\frac{b}{2a}$.

7. Given $ax^2-bx+c=d$, to find x.

Here, $ax^2-bx=d-c$ *by transposition,*

And $x^2-\frac{b}{a}x=\frac{d-c}{a}$ *by division.*

Also $x^2-\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{d-c}{a}+\frac{b^2}{4a^2}$ *by completing the square;*

And $x-\frac{b}{2a}=\pm\sqrt{\frac{d-c}{a}+\frac{b^2}{4a^2}}$ *by evolution;*

Therefore $x=\frac{b}{2a}\pm\sqrt{\frac{d-c}{a}+\frac{b^2}{4a^2}}$.

8. Given $x^4+2ax^2=b$, to find x.

Here, $x^4+2ax^2+a^2=b+a^2$ *by completing the square,*

And $x^2+a=\sqrt{b+a^2}$ *by evolution;*

Whence $x^2=\sqrt{b+a}-a$,

And consequently $x=\sqrt{:\sqrt{b+a}-a}$.

9. Given $ax^n-bx^{\frac{n}{2}}-c=-d$, to find x.

First, $ax^n-bx^{\frac{n}{2}}=c-d$ *by transposition,*

And $x^n-\frac{b}{a}x^{\frac{n}{2}}=\frac{c-d}{a}$ *by division,*

Also $x^n-\frac{b}{a}x^{\frac{n}{2}}+\frac{b^2}{4a^2}=\frac{c-d}{a}+\frac{b^2}{4a^2}$ *by completing the square,*

And $x^{\frac{n}{2}}-\frac{b}{2a}=\sqrt{\frac{c-d}{a}+\frac{b^2}{2a^2}}$ *by evolution;*

Therefore $x^{\frac{n}{2}} = \frac{b}{2a} \pm \sqrt{\frac{c-d}{a} + \frac{b^2}{4a^2}}$,

And consequently $x = \left|\frac{b}{2a} \pm \sqrt{\frac{c-d}{a} + \frac{b^2}{4a^2}}\right|^{\frac{2}{n}}$

EXAMPLES FOR PRACTICE.

1. Given $x^2-8x+10=19$ to find x. *Ans.* $x=9$.
2. Given $x^2-x-40=170$ to find x. *Ans.* $x=15$.
3. Given $3x^2+2x-9=76$ to find x. *Ans.* $x=5$.
4. Given $\frac{x^2}{2}-\frac{x}{3}+7\frac{1}{5}=20$ to find x.
Ans. $x=5.4093$, &c.
5. Given $x^2+x=a$ to find x.

Ans. $x=\sqrt{a+\frac{1}{4}}-\frac{1}{2}$.

6. Given $x^2+ax-b=c$ to find x.

Ans. $x=\sqrt{c+b+\frac{a^2}{4}}-\frac{a}{2}$.

7. Given $x^2-ax=-b$ to find x.

Ans. $x=\pm\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$.

8. Given $\frac{ax^2}{b}-\frac{cx}{d}+\frac{e}{f}=\frac{g}{h}$ to find x.

Ans. $x=\pm\sqrt{\frac{b^2c^2}{4a^2d^2}+\frac{bg}{ah}-\frac{be}{af}}+\frac{bc}{2ad}$.

9. Given $2x^4-x^2+104=600$ to find x.

Ans. $x=4$.

10. Given $3x^n-2x^{\frac{n}{2}}-\frac{p}{9}=\frac{r}{9}$ to find x.

Ans. $x=\left|\pm\sqrt{\frac{r+p+3}{27}}+\frac{1}{3}\right|^{\frac{2}{n}}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whose difference is 8, and product 240.

Let $x =$ *to the least number,*
then will $x+8 =$ *to the greater,*
and $x \times \overline{x+8} = x^2+8x = 240$ *by the question;*
Whence $x^2+8x+16 = 240+16 = 256$ *by completing the square,*
also $x+4 = \sqrt{256} = 16$ *by evolution;*
And therefore $x = 16-4 = 12 =$ *lesser number, and* $12+8 = 20 =$ *greater.*

2. To divide the number 60 into two such parts that their product may be 864.

Let $x =$ *greater part,*
then will $60-x =$ *lesser,*
and $x \times \overline{60-x} = 60x-x^2 = 864$ *by the question,*
that is $x^2-60x = -864$;
Whence $x^2-60x+900 = -864+900 = 36$ *by completing the square;*
Also $x-30 = \sqrt{36} = 6$ *by extracting the root;*
and therefore $x = 6+30 = 36 =$ *greater part,*
and $60-x = 60-36 = 24 =$ *lesser.*

3. Given the sum of two numbers $= 10$ (a), and the sum of their squares $= 58$ (b); to find those numbers.

Let $x =$ *greater of the two numbers,*
then will $a-x =$ *lesser;*
And $x^2 + \overline{a-x}|^2 = 2x^2+a^2-2ax = b$ *by the question,*

or $x^2+\frac{a^2}{2}-ax=\frac{b}{2}$ *by division,*

or $x^2-ax=\frac{b}{2}-\frac{a^2}{2}=\frac{b-a^2}{2}$ *by transposition.*

Whence $x^2-ax+\frac{a^2}{4}=\frac{b-a^2}{2}+\frac{a^2}{4}=\frac{2b-a^2}{4}$ *by completing the square.*

Also $x-\frac{a}{2}=\sqrt{\frac{2b-a^2}{4}}$ *by extracting the root;*

And therefore $x=\pm\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ *greater number,*

And $\overline{a-\frac{a}{2}\pm\sqrt{\frac{2b-a^2}{4}}}=\mp\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ *lesser.*

Hence these two theorems being put into numbers give 7 and 3, for the numbers required.

4. Sold a piece of cloth for 24l. and gained as much *per cent.* as the cloth cost me; what was the price of the cloth?

Let $x=$ *pounds the cloth cost,*
Then $24-x=$ *whole gain,*
But $100:x::x:\underline{24-x}$ *by the question,*
Or $x^2=100\times24-x=2400-100x$,
That is $x^2+100x=2400$,
Whence $x^2+100x+2500=2400+2500=4900$ *by completing the square,*
And $x+50=\sqrt{4900}=70$ *by extraction of roots,*
Consequently $x=70-50=20=$ *price of the cloth.*

5. A person bought a number of oxen for 80l. and if he had bought 4 more for the same money, he would have paid 1l. less for each; how many did he buy?

Suppose he bought x oxen,

Then $\frac{80}{x}$ = *price of each,*

And $\frac{80}{x+4}$ = *price of each if* $x+4$ *had cost* 80l.

But $\frac{80}{x}=\frac{80}{x+4}+1$ *by the question,*

Or $80=\frac{80x}{x+4}+x$,

Or $80x+320=80x+x^2+4x$,

That is $x^2+4x=320$,

Whence $x^2+4x+4=320+4=324$ *by completing the square,*

And $x+2=\sqrt{324}=18$ *by evolution,*

Consequently $x=18-2=16$ = *number of oxen required.*

6. What two numbers are those, whose sum, product, and difference of their squares are all equal to each other?

Let x = *greater number,*

And y = *lesser,*

Then $\left\{\begin{array}{l} x+y=xy \\ x+y=x^2-y^2 \end{array}\right\}$ *by the question,*

And $1=\frac{x^2-y^2}{x+y}=x-y$, *or* $x=y+1$ *from the 2d. equation.*

Also $\overline{y+1}+y=\overline{y+1}\times y$ *from the first equation,*

Or $2y+1=y^2+y$,

That is $y^2-y=1$,

Whence $y^2-y+\frac{1}{4}=1\frac{1}{4}$ *by completing the square,*

Also $y-\frac{1}{2}=\sqrt{1\frac{1}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$ *by evolution,*

Consequently $y=\frac{\sqrt{5}}{2}+\frac{1}{2}=\frac{\sqrt{5}+1}{2}$, *and* $x=y+1$ $=\frac{\sqrt{5}+3}{2}$.

7. There are four numbers in arithmetical progression, whereof the product of the two extremes is 45, and that of the means 77; what are the numbers?

Let $x=$ *lesser extreme,*
and $y=$ *common difference,*
Then x, $x+y$, $x+2y$, $x+3y$ *will be the four numbers,*
and $\left\{\begin{array}{l} x\times\overline{x+3y}=x^2+3xy=45 \\ \overline{x+y}\times\overline{x+2y}=x^2+3xy+2y^2=77 \end{array}\right\}$ *by the question,*

Whence $2y^2=77-45=32$, *and* $y^2=\frac{32}{2}=16$ *by subtraction and division,*
Or $y=\sqrt{16}=4$ *by evolution,*
Therefore $x^2+3xy=x^2+12x=45$ *by the 1st. equation,*
Also $x^2+12x+36=45+36=81$ *by completing the square,*
And $x+6=\sqrt{81}=9$ *by the extraction of roots,*
Consequently $x=9-6=3$, *and the numbers are* 3, 7, 11, *and* 15.

8. To find three numbers in geometrical progression, whose sum shall be 14, and the sum of their squares 84.

Let x, y, *and* z *be the numbers sought,*
Then $xz=y^2$ *by the nature of proportion,*
and $\left\{\begin{array}{l} x+y+z=14 \\ x^2+y^2+z^2=84 \end{array}\right\}$ *by the question,*
But $x+z=14-y$ *by the 2d. equation,*

and $x^2+2xz+z^2=196-28y+y^2$ *by squaring both sides,*
Or $x^2+z^2+2y^2=196-28y+y^2$ *by putting* $2y^2$ *for its equal* $2xz$,
That is $x^2+z^2+y^2=196-28y$ *by subtraction,*
or $196-28y=84$ *by equality,*

Hence $y=\dfrac{196-84}{28}=4$ *by transposition and division.*

Again, $xz=y^2=16$, *or* $x=\dfrac{16}{z}$ *by the 1st equation.*

And $x+y+z=\dfrac{16}{z}+4+z=14$ *by the 2d equation,*

Or $16+4z+z^2=14z$, *or* $z^2-10z=-16$,
Whence $z^2-10z+25=25-16=9$ *by completing the square,*
And $z-5=\sqrt{9}=3$, *or* $z=3+5=8$,
Consequently $x=14-y-z=14-4-8=2$, *and the numbers are* 2, 4, 8.

9. The sum (s) and the product (p) of any two numbers being given; to find the sum of the squares, cubes, biquadrates, &c. of those numbers.

Let the two numbers be denoted by x *and* y,
Then will $\left\{\begin{array}{r} x+y=s \\ xy=p \end{array}\right\}$ *by the question.*
But $\overline{x+y}|^2=x^2+2xy+y^2=s^2$ *by involution,*
and $x^2+2xy+y^2-2xy=s^2-2p$ *by subtraction,*
that is $x^2+y^2=s^2-2p=$ *sum of the squares.*
Again, $\overline{x^2+y^2}\times\overline{x+y}=s^2-2p\times s$ *by multiplication,*
or $x^3+xy\times\overline{x+y}+y^3=s^3-2sp$,
or $x^3+sp+y^3=s^3-2sp$ *by substituting* sp *for its equal* $xy\times\overline{x+y}$;
and therefore $x^3+y^3=s^3-3sp=$ *sum of the cubes.*

In like manner, $\overline{x^3+y^3}\times\overline{x+y}=\overline{s^3-3sp}\times s$ *by multiplication,*

or $x^4+xy\times\overline{x^2+y^2}+y^4=s^4-3s^2p,$

or $x^4+p\times\overline{s^2-2p}+y^4=s^4-3s^2p$ *by substituting* $p\times \overline{s^2-2p}$ *for its equal* $xy\times\overline{x^2+y^2}$;

And consequently, $x^4+y^4=s^4-3s^2p-p\times\overline{s^2-2p}=s^4-4s^2p+2p^2=$ *sum of the biquadrates, or fourth powers; and so on for any power whatever.*

10. The sum (a) and the sum of the squares (b) of four numbers, in geometrical progression, being given; to find those numbers.

Let x *and* y *denote the two means,*

then will $\frac{x^2}{y}$ *and* $\frac{y^2}{x}$ *be the two extremes, by the nature of proportion.*

Also, let the sum of the two means $=s$, *and their product* $=p$.

And then will the sum of the two extremes $=a-s$ *by the question; and their product* $=p$, *by the nature of proportion.*

$$\text{Hence}\left\{\begin{array}{l} x^2+y^2=s^2-2p \\ \frac{x^4}{y^4}+\frac{y^4}{x^2}=\overline{a-s}|^2-2p \end{array}\right\} \text{by the last problem,}$$

And $x^2+y^2+\frac{x^4}{y^2}+\frac{y^4}{x^2}=s^2+\overline{a-s}|^2-4p=b$ *by the question.*

Again, $\frac{x^2}{y}+\frac{y^2}{x}=a-s$ *by the question,*

or $x^3+y^3=xy\times\overline{a-s}=p\times\overline{a-s}.$

But $x^3+y^3=s^3-3sp$ *by the last problem;*

and therefore $p\times\overline{a-s}=s^3-3sp$ *by equality,*

or $pa-ps+3ps=pa+2ps=s^3,$

or $p=\frac{s^3}{a+2s}$;

Whence $s^2+\overline{a-s}|^2-4p=s^2+\overline{a-s}|^2-\frac{4s^3}{a+2s}=b$, *by substitution,*

or $s^2+\frac{b}{a}s=\frac{a^2-b}{2}$ *by reduction.*

And $s=\sqrt{\frac{a^2-b}{2}+\frac{b^2}{4a^2}}-\frac{b}{2a}$ *by comp. the square, and extracting the root.*

And from this value of s, *all the rest of the quantities* p, x, *and* y *may be readily determined.*

QUESTIONS FOR PRACTICE.

1. What two numbers are those whose sum is 20, and their product 36? *Ans.* 2 *and* 18.

2. To divide the number 60 into two such parts, that their product may be to the sum of their squares in the ratio of 2 to 5. *Ans.* 20 *and* 40.

3. The difference of two numbers is 3, and the difference of their cubes is 117: what are those numbers? *Ans.* 2 *and* 5.

4. A company at a tavern had 8 l. 15 s. to pay for their reckoning; but, before the bill was settled, two of them sneaked off, and then those who remained had 10 s. a-piece more to pay than before: how many were there in company? *Ans.* 7.

6. A grazier bought as many sheep as cost him 60 l. and, after reserving 15 out of the number, he sold the remainder for 54 l. and gained 2 s. a-head by them; how many sheep did he buy? *Ans.* 75.

7. There are two numbers whose difference is 15, and half their product is equal to the cube of the lesser number; what are those numbers?

Ans. 3 *and* 18.

8. A person bought cloth for 33 l. 15 s. which he sold again at 2 l. 8 s. *per* piece, and gained by the bargain as much one piece cost him; required the number of pieces? *Ans.* 15.

9. What number is that, which when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? *Ans.* 24.

10. What two numbers are those, whose sum multiplied by the greater is equal to 77; and whose difference multiplied by the lesser is equal to 12? *Ans.* 4 *and* 7.

11. When will the hour, minute, and second hands of a clock be all together next after 12 o'clock? *Ans. Only at* 12 *o'clock.*

12. The sum of two numbers is 8, and the sum of their cubes is 152; what are the numbers? *Ans.* 3 *and* 5.

13. The sum of two numbers is 7, and the sum of their 4th powers is 641; what are the numbers? *Ans.* 2 *and* 5.

14. The sum of two numbers is 6, and the sum of their 5th powers is 1056; what are the numbers? *Ans.* 2 *and* 4.

15. The sum of four numbers in arithmetical progression is 56, and the sum of their squares is 864; what are the numbers? *Ans.* 8, 12, 16, *and* 20.

16. To find four numbers in geometrical progression whose sum is 15, and the sum of their squares 85? *Ans.* 1, 2, 4, *and* 8.

17. Given $\overline{x^2-\frac{a^4}{x^2}}\Big|^{\frac{1}{2}}+\overline{a^2-\frac{a^4}{x^2}}\Big|^{\frac{1}{2}}=\frac{x^2}{a}$, to find the value of x.

Ans. $x=\overline{\tfrac{1}{2}a^2+\sqrt{\frac{5a^4}{4}}}\Big|^{\frac{1}{2}}$.

Of the NATURE and FORMATION of EQUATIONS in general.

All equations of superior orders are generated by the multiplication of equations of inferior orders, involving the same unknown quantity.

Thus, *a quadratic equation* is formed by the multiplication of two simple equations.

A cubic equation is produced by the continued multiplication of three simple equations; or from one quadratic and one simple equation.

A biquadratic equation is generated by the continued multiplication of four simple equations, or of two quadratic equations; or of one cubic and one simple equation, &c.

For, suppose the unknown quantity to be x, and it values in any simple equation to be a, b, c, d, &c.

Then those simple equations, by bringing all the terms to one side, will become $x-a=o$, $x-b=o$, $x-c=o$, $x-d=o$, &c.

And the product of any two of these, as $\overline{x-a}\times\overline{x-b}=o$, will give a *quadratic equation*, or one of two dimensions.

The product of any three of them, as $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=o$, will give a *cubic equation*, or one of three dimensions.

The product of any four of them, as $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=o$, will give a biquadratic equation, or one of four dimensions, &c.

From hence it appears, that every equation has as many roots as it has simple equations that produce it, or as there are units in the highest dimension of the unknown quantity.

For if any of the values of x (a, b, c, *or* d,) be substituted in the place of x in this biquadratic equation $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}$, then will all the terms of that equation vanish, and the whole will be found equal to nothing. But as there are no other quantities, besides these four, that substituted in the place of x will made the product vanish; therefore the equation cannot possibly have more than four roots, or admit of more than four solutions.

And after the same manner it appears, that no equation can have more roots than it contains dimensions of the unknown quantity.

To make this still plainer, by an example in numbers; suppose the equation to be resolved be $x^4-10x^3+35x^2-50x+24=0$, and that you discover that this equation is the same with the product of $\overline{x-1}\times\overline{x-2}\times\overline{x-3}\times\overline{x-4}$.

Then you will certainly infer that the four values of x are 1, 2, 3, and 4; for any of these numbers being put for x will make that product, and consequently $x^4-10x^3+35x^2-50x+24$, equal to nothing, according to the proposed equation.

And it is certain that there can be no other values of x besides these four; since if any other number be substituted for x in those factors, there will none of them vanish, and therefore their product cannot be equal to nothing, according to the equation.

The roots of equations are either *positive* or *negative*, according as the roots of the simple equations from whence they are produced are positive or negative.

Thus, if you suppose $x=-a$, $x=b$, $x=-c$, and $x=d$, then will $x+a=0$, $x-b=0$, $x+c=0$, and $x-d=0$, and the equation $\overline{x+a}\times\overline{x-b}\times\overline{x+c}\times\overline{x-d}=0$ will have its roots $-a$, $+b$, $-c$, $+d$.

The *signs*, and *co-efficients* of equations will be best understood by considering the following table; where the simple equations $x-a$, $x-b$, &c. are multiplied continually together, and produce, successively, the higher equations.

$$\begin{array}{l}
x-a=0 \\
x-b=0 \\
\hline
\left.\begin{array}{l} x^2-ax \\ \quad -bx+ba \end{array}\right\} =0, \text{ a quadratic,} \\
\hline
\times x-c=0 \\
\hline
\left.\begin{array}{r} x^3-a \\ -b \\ -c \end{array}\right\} x^2 \left.\begin{array}{r} +ab \\ +ac \\ +bc \end{array}\right\} x-abc=0, \text{ a cubic,} \\
\hline
\times x-d=0 \\
\hline
\left.\begin{array}{r} x^4-a \\ -b \\ -c \\ -d \end{array}\right\} x^3 \left.\begin{array}{r} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\} x^2 \left.\begin{array}{r} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\} x+abcd=0, \text{ a biquadratic.} \\
\hline
\end{array}$$

&c.

From the inspection of these equations it is plain, that the co-efficient of the first term is *unity*.

The co-efficient of the second term is *the sum of all the roots* (a, b, c, d,) *with contrary signs*.

The co-efficient of the third term is *equal to the sum of the rectangles of those roots, or of all the products that can possibly arise by combining them, two and two.*

The co-efficient of the fourth term *is equal to the sum of all the products that can possibly arise by combin-*

ing them, three and three; and so on for any other co-efficient whatever.

The last term *is always equal to the product of all the roots with contrary signs*; and this reasoning will hold, whether the roots be positive or negative.

It likewise appears, from inspection, that the signs of all the terms of any equation in the table are alternately $+$ and $-$.

The first term is always some pure power of x, and is positive.

The second term is some power of x, multiplied by the quantities $-a$, $-b$, $-c$, &c. and since these quantities are all negative, it follows that the term itself must be negative also.

The third term has the product of any two of these quantities ($-a$, $-b$, $-c$,) for its co-efficient, and is therefore positive; since $-\times-$, as well as $+\times+$, gives $+$, or an affirmative quantity.

For the same reason, the next co-efficient, which is formed of the products of any three of these negative quantities, must be negative; and the next following co-efficient, being made up of the products of any four of the said negative quantities, must be positive, and so on.

And from this reasoning it plainly appears that *when all the roots are positive, the signs are plus and minus alternately.*

But if the roots are all negative ($x=-a$, $x=-b$, $x=-c$, $x=-d$) then $\overline{x+a}\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}=0$, will express the equation to be produced; and all the terms will, plainly, be positive.

So that, *when all the roots of an equation are negative, it is plain that there can be no changes in the signs of the terms of that equation.*

And, in general, there will be as many positive roots in any equation, as there are changes in the

ſigns of the terms of that equation, from + to —, or from — to +; and all the reſt of the roots will be negative.

From this rule it follows that, in quadratic equations, the two roots may be either both poſitive, or both negative, or one negative and one affirmative.

Thus, in the equation, $x^2 \begin{smallmatrix}-ax\\-bx\end{smallmatrix} + ab = o$, $(\overline{x-a} \times \overline{x-b})$ there are two changes of the ſigns, and therefore the roots are both poſitive.

In the equation $x^2 \begin{smallmatrix}+ax\\+bx\end{smallmatrix} + ab = o$, $(\overline{x+a} \times \overline{x+b})$ there is no change of the ſigns, and conſequently they are both negative.

And, in this equation, $x^2 \begin{smallmatrix}-ax\\-bx\end{smallmatrix} + ab = o$, $(\overline{x-a} \times \overline{x+b})$ one of the roots will be affirmative and one negative; for ſince the firſt term is poſitive and the laſt negative, there can be but one change in the ſigns, whether the ſecond term be + or —.

In cubic equations, the roots may be all poſitive, or all negative; or two of them may be negative and one poſitive, or one negative and two poſitive.

Thus, in the equation $\overline{x-a} \times \overline{x-b} \times \overline{x-c} = o$, the ſigns will be alternately + and —; and, as the number of changes is three, the roots muſt be all poſitive.

In the equation $\overline{x+a} \times \overline{x+b} \times \overline{x+c} = o$, where there are no changes of the ſigns, the roots muſt be all negative.

In the equation $x^3 - \overline{a-b+c}\,x^2 + \overline{ab-ac-bc}\,x + abc = o$, $(\overline{x-a} \times \overline{x-b} \times \overline{x+c})$ the number of changes will be two, and conſequently two of the roots will be affirmative and one negative.

For if $a+b$ is greater than c, the ſecond term muſt be negative, its co-efficient being $-a$, $-b$, $+c$;

and if $a+b$ is leſs than c, the third term muſt be negative, its co-efficient $+ab-ac-bc$ $(ab-c\times\overline{a+b})$ being in that caſe negative.

In the equation $x^3+\overline{a+b-c}x^2+\overline{ab-ac-bc}x-abc=o$, there can be only one change of the ſigns, and therefore one of the roots is poſitive, and the other two negative.

For if $a+b$ is leſs than c then the ſecond term is negative, and the third muſt be negative alſo; and if $a+b$ is greater than c the ſecond term will be poſitive, and there can be but one change in the other two terms, whatever may be their ſigns.

And, in the ſame manner, this reaſoning may be extended to equations of higher dimenſions, and therefore the rule will be found to be true in all kinds of equations whatever.

PROBLEM I.

To increaſe or diminiſh the roots of an equation by any given quantity.

RULE.

1. Take ſome new letter, and connect it with the given quantity by the ſigns — or +, according as it is required to be increaſed or diminiſhed.

2. Subſtitute the powers of this quantity in the equation, inſtead of the powers of the unknown letter, and there will ariſe a new equation, whoſe roots will be augmented or diminiſhed as required.

EXAMPLES.

1. Let the quadratic equation $x^2-8x+15=0$, be given; it is required to increase its roots by 7.

Suppose $x=y-7$,

Then $x^2=y^2-14y+49$

$8x= \quad +8y-56$

$15= \quad\quad +15$

$y^2-6y+8=0=$ *equal to the equation required* *.

2. Let $x^3-px^2+qx-r=0$, be the equation given; it is required to diminish the roots by the quantity e.

Suppose $x=y+e$;

$$\left.\begin{array}{l} \textit{Then } x^3=y^3+3ey^2+3e^2y+e^3 \\ -px^2=\quad -py^2-2pey-pe^2 \\ +qx= \quad\quad +qy\ +qe \\ -r= \quad\quad\quad -r \end{array}\right\}=0, \textit{ the new equation required } \dagger.$$

3. Let $x^3+x^2-10x+8=0$ be given, and let its roots be increased by 4.

---

* For in the former equation $x^2+8x+15=0$, the roots are -3 and -5, and in the equation $y^2-6y+8=0$, the roots are 2 and 4; and therefore the difference is 7, as required.

† The last term of this transformed equation, is the same as the equation given, having e in the place of y.

Suppose $x=y-4$;
Then $x^3=y^3-12y^2+48y-64$
$+\ x^2=\ +\ y^2-\ 8y+16$
$-10x=\ \ -10y+40$
$+8=\ \ +8$

$y^3-11y^2+30y=0$, *or* $y^2-11y+30=0$, *the equation required* ‡.

PROBLEM II.

To take away the second term from any equation.

RULE.

1. Divide the co-efficient of the second term by the index of the higheſt power of the unknown quantity.

2. Annex the quotient, with its ſign changed, to ſome new letter, and this being ſubſtituted for its equal in the given equation, will deſtroy the ſecond term, as required.

‡ In this example the given equation is reduced to a quadratic, and in the preſent caſe, as well as in all others where the laſt term vaniſhes, the number aſſumed (-4) is one of the roots of the propoſed equation.

The affirmative roots of an equation are changed into negative ones of the ſame value, and the negative roots into affirmative ones, by only changing the ſigns of the terms alternately, beginning with the ſecond.

PROBLEM III.

To find whether some, or all the roots of an equation be rational; and, if so, what they are.

RULE.*

1. Find all the divisors of the last term, and substitute them one by one for the unknown quantity.

2. Then, if the positive and negative terms destroy each other, the divisor, so substituted, will be one of the roots of the equation.

3. But, if none of the divisors succeed, the roots are, for the general part, either irrational or impossible.

Note, When the divisors of the last term are too numerous, they may be diminished by changing the equation into another, whose roots are augmented or decreased by an unit, or some other known quantity.

EXAMPLES:

1. Let $x^3-4x^2-7x+10=0$ be the equation proposed.

Then the divisors of (10) *the last term will be* $+1$, -1, $+2$, -2, $+5$, -5, $+10$, -10.

* Since the last term, in any equation, is always equal to a multiple of all the roots in that equation, those roots must, therefore, necessarily be found in the number of its divisors.

And these being substituted successively instead of x, will give

$$\begin{array}{r}
1-4-7+10=0 \\
-1-4+7+10=12 \\
8-16-14+10=-12 \\
-8-16+14+10=0 \\
125-100-35+10=0
\end{array}$$

Therefore $+1-2$ *and* $+5$ *are the three roots of the equation required.*

2. Let $y^4-4y^3-8y+32=0$, be the equation proposed.

1. *Change it into another, the number of whose divisors shall be less; thus:*

Suppose $y=x+1$;

Then

$$\begin{array}{lr}
y^4= & x^4+4x^3+6x^2+4x+1 \\
-4y^3= & -4x^3-12x^2-12x-4 \\
-8y= & -8x-8 \\
+32= & +32 \\
\hline
\end{array}$$

$x^4-6x^2-16x+21=0=$ *new equation.*

2. *The divisors of the last term* (21) *of this new equation are*

$1, -1, +3, -3, +7, -7, +21, -21.$

And if these be substituted successively instead of x, we shall have

$$\begin{array}{r}
1-6-16+21=0 \\
1-6+16+21=32 \\
81-54-48+21=0 \\
81-54+48+21=96
\end{array}$$

&c. where none of the others succeed.

So that 1 *and* 3 *are the only rational roots, the other two being impossible.*

3. Let $x^3+3ax^2-4a^2x-12a^3=0$, be the equation proposed.

Here the numeral divisors of the last term $(12a^3)$ *are*

$1, -1, +2, -2, +3, -3, +4, -4, +6, -6, +12, -12.$

And by substituting these successively instead of x, we shall have

$$1+3-4-12=-12$$
$$-1+3+4-12=-6$$
$$8+12-8-12=0$$
$$-8+12+8-12=0$$
$$27+27-12-12=30$$
$$-27+27+12-12=0$$

Therefore the three roots are 2*a*, —2*a and* 3*a*.

PROBLEM IV.

To discover the roots of equations by SIR ISAAC NEWTON's *method of divisors.*

RULE.

1. Instead of the unknown quantity, substitute successively three, or more, terms of the arithmetical progression 2, 1, 0, —1, —2.

2. Collect all the terms of the equation into one sum, and place them, together with their divisors, in a perpendicular line, right against the corresponding terms of the progression 2, 1, 0, —1, —2.

3. Seek amongst the divisors for an arithmetical progression whose terms correspond with the order of the terms 2, 1, 0, —1, —2, and whose common difference is either an unit, or some divisor of the co-efficient of the highest power of the unknown quantity in the given equation.

4. Divide that term of the progression, thus found, which stands against the term 0 in the first progression by the ratio or common difference.

5. To the quotient, last found, prefix the sign + or —, according as the progression is increasing or

decreasing, and this number being substituted for the unknown quantity, will give one of the roots of the equation.

EXAMPLES:

1. Let $x^3-x^2-10x+6=0$, be the equation proposed.

Then, by substituting successively the terms of the progression 2, 1, 0, —1, *instead of* x, *the work will stand as follows:*

1st prog.	*results.*	*divisors.*	*2d prog.*
2	—10	1 . 2 . 5 . 10	5
1	— 4	1 . 2 . 4	4
0	+ 6	1 . 2 . 3 . 6	3
—1	+ 14	1 . 2 . 7 . 14	2

And —3, *the term standing against* 0, *being substituted for* x, *gives* $-27-9+30+6=0$; *and therefore* —3 *is a root of the equation.*

2. Let $2x^3-5x^2+4x-10=0$, *be the equation proposed.*

Then, by substituting successively the terms of the progression 2, 1, 0, —1, —2, *instead of* x, *the work will stand as follows:*

1st prog.	*results.*	*divisors.*	*2d. prog.*
2	— 6	1.2.3. 6	1
1	— 9	1.3.9	3
0	—10	1.2.5.10	5
—1	—21	1.3.7.21	7
—2	—54	1.2.3. 6.9	9

Here 5, *the term standing against* 0, *being divided by* 2, *the common difference, gives* $2\frac{1}{2}$; *and this being substituted for* x, *gives* $31\frac{1}{4}-31\frac{1}{4}+10-10=0$; *and therefore* $2\frac{1}{2}$ *is a root of the equation.*

3. Let $x^4+x^3-29x^2-9x+180=0$, be the equation proposed.

Then, by substituting as before, the work will stand as follows:

1st pro.	results.	divisors.	progressions.			
2	70	1.2.5.7.10.14 &c.	1	2	5	7
1	144	1.2.3.4. 6. 8 &c.	2	3	4	6
0	180	1.2.3.4. 5. 6 &c.	3	4	3	5
-1	160	1.2.4.5. 8.10 &c.	4	5	2	4
-2	90	1.2.3.5. 6. 9 &c.	5	6	1	3

So that here are four progressions, and the numbers 3, 4, -3, and -5, being each substituted for x, make the whole equation vanish, and are therefore the roots required.

PROBLEM V.

To find the roots of cubic equations, according to the method of CARDAN.

RULE.*

1. Take away the second term of the equation, by problem 2d, and it will be reduced to this form $x^3 \pm ax = \pm b$.

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* The rule, from whence this method is derived, is $x = \sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}} + \sqrt[3]{\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, and the investigation of it is as follows:

Let the equation, whose root is required, be $x^3+ax=b$.

Put $y+z=x$, and $3yz=-a$.

Then, by substituting these values in the given equation, we shall have, $y^3+3y^2z+3yz^2+z^3+a\times\overline{y+z}=y^3+z^3+3yz\times\overline{y+z}+a\times\overline{y+z}=y^3+z^3-a\times\overline{y+z}+a\times\overline{y+z}=y^3+z^3=b$.

2. Substitute the values of a and b, with their proper signs, in the following expression, and it will give the root required. Thus:

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=x=$$

root required.

Note, When a is negative, and $\frac{1}{27}a^3$ is greater than $\frac{1}{4}b^2$, the equation, by this rule, is generally impossible.

And if, from the square of this last equation, there be taken 4 times the cube of the equation $yz=-\frac{1}{3}a$, we shall have $y^6-2y^3z^3+z^6=b+\frac{4}{27}a^3$, or $y^3-z^3=\sqrt{b+\frac{4}{27}a^3}$.

But the sum of this equation and $y^3+z^3=b$, is $2y^3=b+\sqrt{b+\frac{4}{27}a^3}$; and their difference is $2z^3=b-\sqrt{b+\frac{4}{27}a^3}$; and therefore $y=\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, and $z=\sqrt[3]{\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$.

And from hence it appears, that $y+z$, or its equal x, is $=\sqrt[3]{\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}+\sqrt[3]{\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$, which is the theorem.

Or, since z is $=-\dfrac{\frac{1}{3}a}{y}$, it will be $y+z=x=$

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}},$$ *the same as the rule.*

This method of solving cubic equations is usually ascribed to *Cardan*; but the invention is, undoubtedly, not his.— The authors of it were *Scipio Ferreus* and *Nicholas Tartalca*, who discovered it about the same time, independently of each other, as is proved by *M. de Montucla* dans son *Histoire des Mathématiques*.

EXAMPLES:

1. Let $y^3+3y^2+9y=13$, be the equation proposed; it is required to find the value of y.

1. *In order to destroy the second term, let* $y=x-1$; *then*

$$\begin{aligned} y^3 &= x^3-3x^2+3x-1 \\ 3y^2 &= \quad +3x^2-6x+3 \\ 9y &= \qquad\quad +9x-9 \\ \hline & x^3+6x-7=13 \\ \text{or } & x^3+6x=20. \end{aligned}$$

2. *For a put* 6, *and for b* 20, *and we shall have*

$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=$$

$$\sqrt[3]{10+\sqrt{100+8}}-\frac{2}{\sqrt[3]{10+\sqrt{100+8}}}=$$

$$\sqrt[3]{10+10.3923}-\frac{2}{\sqrt[3]{10+10.3923}}=\sqrt[3]{20.3923}-$$

$$\frac{2}{\sqrt[3]{20.3923}}=2.732-\frac{2}{2.732}=2.732-.732=2;$$ *that is* $x=2$, *and consequently* $y=1=$ *root required.*

2. Given $x^3-6x=-9$, to find the value of x.

Here $a=-6$, *and* $6=-9$; *and therefore we shall have*

$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=$$

$$\sqrt[3]{-4\tfrac{1}{2}+\sqrt{20\tfrac{1}{4}-8}}-\frac{-2}{\sqrt[3]{-4\tfrac{1}{2}+\sqrt{20\tfrac{1}{4}-8}}}=$$

$$\sqrt[3]{-4\tfrac{1}{2}+3\tfrac{1}{2}}-\frac{-2}{\sqrt[3]{-4\tfrac{1}{2}+3\tfrac{1}{2}}}=\sqrt[3]{-1}-\frac{-2}{\sqrt[3]{-1}}=-1$$

$-\frac{-2}{-1}=-1-2=-3$; *that is* $x=-3=$ *root required.*

EXAMPLES FOR PRACTICE.

1. Given $x^3-5x^2+10x-8=0$, to find x.
Ans. $x=4$.
2. Given $x^3+30x=117$, to find x. *Ans.* $x=3$.
3. Given $y^3-36y=91$, to find y. *Ans.* $y=7$.
4. Given $x^3+6x=30\sqrt{3}$, to find x.
Ans. $x=2\sqrt{3}$.
5. Given $y^3+24y=250$, to find y. *Ans.* $y=5.05$.
6. Given $y^6-3y^4-2y^2-8=0$, to find y.
Ans. $y=2$.

PROBLEM VI.

To find the roots of biquadratic equations, according to the method of DES CARTES.

RULE.*

1. Take away the second term of the equation by problem 2, and it will be reduced to the form $x^4+qx^2+rx+s=0$.

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* *Investigation of the rule.* Let the given equation $x^4+qx^2+rx+s=0$, be equal to the product of the two quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0=x^4+\overline{f+g-e^2}\,x^2+\overline{eg-ef}\,x+fg=0$.

Then, by equating the homologous terms, we shall have $f+g-e^2=q$, $eg-ef=r$, and $fg=s$; and therefore $f=$

2. From the cubic equation $y^3+2qy^2+\overline{q^2-4s}\,y-r^2=0$ take the ſecond term, and find the value of y by the laſt problem.

3. Put $e=\sqrt{y}$, $f=\frac{q}{2}+\frac{e^2}{2}-\frac{r}{2e}$, and $g=\frac{q}{2}+\frac{e^2}{2}+\frac{r}{2e}$.

4. Find the roots of the two quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0$, and they will be the four roots of the biquadratic required.

EXAMPLES:

1. Let $z^4-4z^3-8z+32=0$, be the equation propoſed, in which it is required to find the value of z.

$\frac{q}{2}+\frac{e^2}{2}-\frac{r}{2e}$, $g=\frac{q}{2}+\frac{e^2}{2}+\frac{r}{2e}$, and $s=f\times g=\frac{q^2}{4}+\frac{qe^2}{2}+\frac{e^4}{4}-\frac{r^2}{4e^2}$.

And from this laſt equation we ſhall have $e^6+2qe^4+\overline{q^2-4s}\times e^2=r^2=$ to a cubic equation, in which the value of e may be found, as in the laſt problem.

But $f\left(=\frac{q}{2}+\frac{e^2}{2}-\frac{r}{2e}\right)$ and $g\left(=\frac{q}{2}+\frac{e^2}{2}+\frac{r}{2e}\right)$ are alſo known; and therefore the roots of the quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0$, may be determined, and are the four roots of the biquadratic equation required. Q. E. I.

Note, The co-efficient of x is put equal to e, in both the equations, becauſe when the ſecond term is wanting, the ſum of the poſitive roots is always equal to the ſum of the negative ones, with a contrary ſign.

This rule has ſometimes been aſcribed to *Des Cartes*, and ſometimes to *Bombelli*, an Italian; but the original inventor of it was *Louis Ferrari*.

1. *To take away the second term, let* $x+1=z$; *then*

$$\begin{aligned} z^4 &= x^4+4x^3+6x^2+4x+1 \\ -4z^3 &= \quad -4x^3-12x^2-12x-4 \\ -8z &= \quad -8x-8 \\ +32 &= \quad +32 \\ \hline & x^4-6x^2-16x+21=0, \text{ or} \end{aligned}$$

$y^3-12y^2-48y-256=0$, *for the cubic equation.*

2. *To take away the second term from this equation, let* $p+4=y$; *then*

$$\begin{aligned} y^3 &= p^3+12p^2+48p+64 \\ -12y^2 &= \quad -12p^2-96p-192 \\ -48y &= \quad -48p-192 \\ -256 &= \quad -256 \\ \hline & p^3-96p=576 \end{aligned}$$

The following method of resolving biquadratic equations, is taken from Simpson's Algebra, page 150, 2*d edition.*

In the method of *Des Cartes*, above explained, all biquadratic equations are supposed to be generated from the multiplication of two quadratic ones: but according to the way which I am now going to lay down, every such equation is conceived to arise by taking the difference of two complete squares.

Here, the general equation $x^4+ax^3+bx^2+cx+d=0$ being proposed, we are to assume $\overline{x^2+\frac{1}{2}ax+A}|^2-\overline{Bx+C}|^2$ $=x^4+ax^3+bx^2+cx+d$; in which A, B, and C, represent unknown quantities to be determined.

Then, $x^2+\frac{1}{2}ax+A$, and $Bx+C$ being actually involved, will give $\left.\begin{array}{l} x^4+ax^3+2Ax^2 \\ \quad +\frac{1}{4}a^2x^2+aAx+A^2 \\ \quad -B^2x^2-2BCx-C^2 \end{array}\right\} = x^4+ax^3+bx^2$ $+cx+d$: from whence, by equating the homologous terms we shall have

3. *To find the value of p, by* CARDAN'S *rule for cubic equations.*

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=$$

$$\sqrt[3]{288+\sqrt{288^2-32^3}}-\frac{-32}{\sqrt[3]{288+\sqrt{288^2-32^3}}}=$$

$12=p$; *and therefore* $y=16$, *or* $\sqrt{y}=4$, $f=\frac{-6}{2}+\frac{16}{2}+\frac{16}{8}=7$, *and* $g=\frac{-6}{2}+\frac{16}{2}-\frac{16}{8}=3$.

4. *To find the roots of the two quadratic equations* $x^2+ex+f=0$, *and* $x^2-ex+g=0$.

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1. $2A+\frac{1}{4}a^2-B^2=b$, or $2A+\frac{1}{4}a^2-b=B^2$;
2. $aA-2BC=c$, or $aA-c=2BC$;
3. $A^2-C^2=d$, or $A^2-d=C^2$.

Let now the first and last of these equations be multiplied together, and the product will, evidently, be equal to $\frac{1}{4}$ of the square of the second; that is $2A^3+\overline{\frac{1}{4}a^2-b}\times A^2-2dA-\overline{\frac{1}{4}a^2-b}\times d\,(=B^2C^2)=\frac{1}{4}\times\overline{a^2A^2-2acA+c^2}$.

Whence, by denoting the given quantities $\frac{1}{4}ac-d$, and $\frac{1}{4}c^2+d\times\overline{\frac{1}{4}a^2-b}$ by k and l, respectively, there will arise this cubic equation, $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=0$; by means whereof, the value of A may be determined; and therefore, from the preceding equations, both B and C will also be known, B being found from thence $=\sqrt{2A+\frac{1}{4}a^2-b}$, and $C=\frac{aA-c}{2B}$.

The several values of A, B, and C being thus found, that of x will also be readily obtained: for $\overline{x^2+\frac{1}{2}ax+A}|^2-\overline{Bx+C}|^2$ being universally, in all circumstances of x, equal to $x^4+ax^3+bx^2+cx+d$, it is evident, that, when the value of x is taken such that the latter of these expressions becomes equal to nothing, the former must likewise be equal to nothing; and consequently $\overline{x^2+\frac{1}{2}ax+A}|^2=\overline{Bx+C}|^2$.

$$x^2+ex+f=x^2+4x+7=0$$
$$x^2-ex+g=x^2-4x+3=0$$

In the first of these $x=-2+\sqrt{-3}$, *or* $-2-\sqrt{-3}$.
And in the second $x=3$, *and* 1.

Therefore 3, 1, $-2+\sqrt{-3}$, *and* $-2-\sqrt{-3}$ *are the four roots of the equation* $x^4-6x^2-16x+21=0$.

And if unity be added to each of them, we shall have 4, 2, $-1+\sqrt{-3}$, *and* $-1-\sqrt{-3}$ *for the roots of* $z^4-4z^3-8z+32=0$, *the equation proposed; the two last of which are impossible.*

2. Given $x^4+2x^3-7x^2-8x+12=0$, to find the values of x. *Ans.* $x=1$, 2, -3 *and* -2.

3. Given $x^4-25x^2+60x-36=0$, to find the values of x. *Ans.* 1, 2, 3 *and* -6.

4. Given $y^4-8y^3+14y^2+4y-8=0$, to find the value of y.

Ans. $y=3+\sqrt{5}$, $3-\sqrt{5}$, $1+\sqrt{3}$, *and* $1-\sqrt{3}$.

And therefore, by extracting the square root of both sides of the equation, we shall have $x^2+\frac{1}{2}ax+A=\pm Bx\pm C$; or $x=\pm\frac{1}{2}B-\frac{1}{4}a\pm\overline{\overline{\tfrac{1}{4}a\pm\tfrac{1}{2}B}^{\,2}\pm C-A}^{\,\frac{1}{2}}=\pm\frac{1}{2}B-\frac{1}{4}a\pm\overline{\tfrac{1}{16}a^2\mp\tfrac{1}{4}aB+\tfrac{1}{4}B^2\pm C-A}^{\,\frac{1}{2}}$; which exhibits all the different roots of the given equation, according to the variation of the signs.

This method will be found to have many advantages over that given above. In the first place, there is no necessity of being at the trouble of exterminating the second term of the equation, in order to prepare it for a solution: secondly, the equation $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=0$, here brought out, is of a more simple form than that derived from the former method: and thirdly, the value of A in this equation, will always be *commensurate* and *rational*; not only when all the roots of the given equation are *commensurate*, but also when they are *irrational*, and even *impossible*.

EXAMPLE. *Let there be given* $x^4+12x-17=0$, *to find the values of* x.

5. Given $x^4+12x-17=0$, to find the values of x.
Ans. $x=\frac{1}{2}\sqrt{2}\pm\sqrt{-3\sqrt{2}-\frac{1}{2}}$, *and* $-\frac{1}{2}\sqrt{2}\pm\sqrt{3\sqrt{2}-\frac{1}{2}}$.

6. Given $x^4-6x^3-58x^2-114x-11=0$, to find the values of x.

Ans. $x=\pm\frac{5}{2}\sqrt{3}+\frac{3}{2}\pm\sqrt{17\pm\frac{21}{2}\sqrt{3}}$.

7. Given $x^4-3x^2-4x-3=0$, to find the values of x. *Ans.* $x=\frac{1\pm\sqrt{3}}{2}$ *and* $\frac{-1\pm\sqrt{-3}}{2}$.

Here, by comparing this with the general equation $x^4+ax^3+bx+cx+d=0$, we ſhall have $a=0$, $b=0$, $c=12$, and $d=-17$; and therefore $k(=\frac{1}{4}ac-d)=17$, $l(=\frac{1}{4}c^2+d\times\overline{\frac{1}{4}a^2-b})=36$, and $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=A^3+17A-18=0$.

And, from this equation, A will be found equal to 1; and therefore $B(=\overline{2A+\frac{1}{4}a^2-b}|^{\frac{1}{2}})=\sqrt{2}$, $C\left(=\frac{aA-c}{2B}\right)=\frac{-12}{2\sqrt{2}}=-3\sqrt{2}$, and $x=\pm\frac{1}{2}\sqrt{2}\pm\overline{\frac{1}{2}\mp3\sqrt{2}-1}|^{\frac{1}{2}}=\pm\frac{1}{2}\sqrt{2}\mp\overline{\mp3\sqrt{2}-\frac{1}{2}}|^{\frac{1}{2}}$.

Hence, the four roots of the equation are $\frac{1}{2}\sqrt{2}+\overline{-3\sqrt{2}-\frac{1}{2}}|^{\frac{1}{2}}$, $\frac{1}{2}\sqrt{2}-\overline{-3\sqrt{2}-\frac{1}{2}}|^{\frac{1}{2}}$, $-\frac{1}{2}\sqrt{2}+\overline{3\sqrt{2}-\frac{1}{2}}|^{\frac{1}{2}}$, and $-\frac{1}{2}\sqrt{2}-\overline{3\sqrt{2}-\frac{1}{2}}|^{\frac{1}{2}}$; the firſt and ſecond of which are impoſſible.

In ſome particular caſes of this rule, the roots may be found by means of quadratics only.

Several other methods of ſolving biquadratic equations have been invented by different authors; but one of the moſt ingenious is that given by M. *Euler*, in page 664 of his *Elémens D' Algebre*.

Equations of five, or more dimenſions, may be reduced to thoſe of an inferior degree; but the proceſs will be exceedingly tedious, as no general rule can be given for reſolving them.

PROBLEM VII.

To find the roots of equations in general, by the methods of approximation and converging series.

RULE.*

1. Find, by trial, a number nearly equal to the root required.

2. Call the number, thus found, r, and let z be put equal to the difference between r and the true root x.

3. Instead of x, in the given equation, substitute its equal $r \pm z$, and there will arise a new equation, affected only with z and known quantities.

4. Reject all those quantities in which there are two or more dimensions of z, and the value of z will be found by means of a simple equation.

5. Add the value of z, thus found, to the value of r, and it will give the root required *nearly*.

6. If this root is not sufficiently near the truth, repeat the operation, by substituting it instead of r, in the equation exhibiting the value of z, and it will give a *second correction* for the root required.

EXAMPLES:

1. Given $x^2-5x-31=0$, to find the value of x by approximation.

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* The rules hitherto given, for finding the roots of equations, are either very troublesome and laborious, or else confined to particular cases; but this method, by converging series, is universal, extending to all kinds of equations whatever; and, though not accurately true, gives the value sought to any assigned degree of exactness.

The method of obtaining the roots of equations, by approximation, was first made use of by *Vieta*.

The root, found by trial, is nearly equal to 8;
Let, therefore, $8=r$, *and* $r+z=x$; *then*

$$x^2=r^2+2rz+z^2$$
$$-5x=-5r-5z$$
$$-31=-31$$

$$r^2+2rz-5r-5z-31=0;$$

or $z=\frac{31+5r-r^2}{2r-5}=\frac{31+40-64}{16-5}=\frac{7}{11}=.6$, *and* $x=$ 8.6 *nearly.*

And, again, if 8.6 *be substituted in the place of* r *in the last equation, we shall have*

$z=\frac{31+5r-r^2}{2r-5}=\frac{31+43-73.96}{17.2-5}=\frac{.04}{12.2}=.0032$
and $x=8.6+.0032=8.6032$ *nearly.*

And, if this value be again substituted for r, *it will give* $z=.000077808$, *and* $x=8.603277808$; *and so on to any degree of exactness required.*

2. Given $x^3+x^2+x=90$, to find the value of x by approximation.

The root, found by trial, is nearly equal to 4;
Let, therefore, $4=r$, *and* $r+z=x$, *then,*

$$x^3=r^3+3r^2z+3rz^2+z^3$$
$$x^2=r^2+2rz+z^2$$
$$x=r+z$$

$$r^3+3r^2z+r^2+2rz+r+z=90;$$

or, $z=\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-64-16-4}{48+8+1}=\frac{6}{57}=.10$,
and $x=4.1$ *nearly.*

And again, if 4.1 *be substituted in the place of* r *in the last equation, we shall have*

$$\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-68.921-16.81-4.1}{50.43+8.2+1}=.00283,$$

and $x=4.1+.00283=4.10283$ *nearly; and ſo on to any degree of exactneſs required.*

3. Given $x^2+20x=100$, to find the value of x by approximation. *Anſ.* $x=4.1421356236$.

4. Given $x^3+10x^2+5x=2600$, to find the value of x by approximation. *Anſ.* 10.1794653.

5. Given $x^3+2x^2-23x-70=0$, to find the value of x. *Anſ.* $x=5.1349$.

6. Given $x^3-15x^2+63x-50=0$, to find the value of x. *Anſ.* $x=1.0280392327$.

7. Given $x^4-3x^2-75x=10000$, to find the value of x. *Anſ.* $x=9.8860027$.

8. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x. *Anſ.* $x=8.1414455$.

RULE II.

1. Aſſume the general equation $az+bz^2+cz^3+dz^4$, &c. $=p$; where z is the converging quantity, and a, b, c, d, &c. co-efficients whoſe values are known.

2. Then will $\frac{ap}{a^2+bp}$ be an approximation of the firſt degree.

3. And, if s be put $=\frac{b}{a}-\frac{c}{b}$ we ſhall have $\frac{\overline{a+sp}\times p}{a^2+\overline{b+as}\times p}$ for an approximation of the ſecond degree.

4. And, in like manner, if w be put $=\frac{2b}{a}+$

$\frac{ad-bc}{b^2-ac}$, then will $\frac{ap \times \overline{a+wp}}{a \times \overline{a^2+b+aw} \times p + \overline{w-s} \times p^2}$ be an approximation of the third degree, &c.

EXAMPLES:

1. Given $x^2+20x=100$, to find the value of x.

The root, found by trial, is nearly equal to 4;

Let therefore $4+z=x$, and, by substitution, the equation will become $28z+z^2=4$.

Whence, from the rule, $a=28$, $b=1$, $c=0$, &c. and $p=4$.

Therefore $z=\frac{ap}{a^2+bp}=\frac{112}{788}=\frac{28}{197}=.14213$ *for the first approximation.*

And, since $s=\frac{b}{a}-\frac{c}{b}=\frac{1}{28}$, *we shall have*

$$z=\frac{\overline{a+sp} \times p}{a^2+\overline{b+as} \times p}=\frac{\overline{28+\frac{1}{7}} \times 4}{28 \times 28+\overline{1+1} \times 4}=\frac{28+\frac{1}{7}}{28 \times 7+2}=$$

$\frac{197}{1386}=.1421356$ *for the second approximation.*

And, in like manner, since $w=\frac{2b}{a}+\frac{ad-bc}{b^2-ac}=\frac{1}{14}$, *it will be* $z=\frac{ap \times \overline{a+wp}}{a \times \overline{a^2+b+aw} \times p+\overline{w-s} \times p^2}=$

$$\frac{28 \times 4 \times \overline{28+\frac{2}{7}}}{28 \times 784+12+\frac{4}{7}}=\frac{28 \times \overline{28+\frac{2}{7}}}{7 \times 796+\frac{1}{7}}=\frac{28 \times 198}{49 \times 796+1}=$$

$\frac{5544}{39005}=.1421356236$; *or* $x=4.1421356236$ *for the root required, extremely near.*

2. Given $x^2+4x-10=100$, to find the value of x. *Anf.* 10.677078.

3. Given $x^3-17x^2+54x=350$, to find the value of x. *Anf.* 14.95407.

4. Given $x^3-2x-5=0$, to find the value of x. *Anf.* 2.09455147.

5. Given $2y^4-16y^3+40y^2-30y=-1$, to find the value of y. *Anf.* $y=1.284724$.

PROBLEM VIII.

To extract the root of any pure power in numbers.

RULE.

1. Let $m=$ the number whofe root is required; $r=$ neareft root which can be found by trial; and $n=$ to the index.

2. Then, by putting $v=\frac{nr^n}{m-r^n}$, we fhall have

$$x=r+\frac{r\times\overline{6v+n+1}}{v\times\overline{6v+4n-2}}=\text{root, } \textit{nearly}\text{; or } x=r+$$

$$\frac{r\times\overline{2v+n}}{v\times\overline{2v+2n-1}+\frac{1}{6}\times\overline{n-1}\times\overline{2n-1}}\ \textit{extremely near.}$$

EXAMPLES:

1. Given $x^2=2$, or, which is the fame thing, let the fquare root of 2 be found.

Suppose the root, found by trial, to be 1.4; *then we shall have* $m=2$, $r=1.4$, $n=2$, *and* $v=\frac{2\times1.96}{2-1.96}=98$.

And, therefore, $x=r+\frac{r\times\overline{6v+n+1}}{v\times\overline{6v+4n-2}}=1.4+\frac{1.4\times591}{98\times594}=1.4+\frac{197}{70\times198}=1.4+\frac{197}{13860}=1.41421356=$ *root required nearly.*

And, if the second approximation be used, the root will be found $=1.41421356236$, *which is true to the last place of decimals.*

2. Given $x^3=500$; or let it be required to extract the cube root of 500.

Suppose the root, found by trial, to be 8; *then we shall have* $m=500$, $r=8$, $n=3$, *and* $v=\frac{3\times512}{-12}=-128$;

And, therefore $x=r+\frac{r\times\overline{6v+n+1}}{v\times\overline{6v+4n-2}}=8-.063=7.93$ *for the 1st approximation.*

Or $x=r+\frac{r\times\overline{2v+n}}{\overline{2v+2n-1}\times\overline{v+\frac{1}{6}}\times\overline{n-1}\times\overline{2n-1}}=8-\frac{6072}{96389}=7.937005259936$ *for the 2d approximation, which is true to the last place of decimals.*

3. Let it be required to find the cube root of 2.

Ans. 1.259921049894.

4. What is the cube root of 117? *Ans* 4.89097.

5. What is the surfolid, or 5th root, of 125000?

Ans. 10.456389.

6. It is required to find the 7th root of 100000.

Ans. 5.179474679.

PROBLEM IX.

To find the root of an exponential equation.

RULE.*

1. Find, by trial, two numbers, as near the true root as poſſible, and ſubſtitute them in the given equation inſtead of the unknown quantity, marking the errors which ariſe from each of them.

2. Multiply the difference of the two numbers, found by trial, by the leaſt error, and divide the product by the difference of the errors, when they are alike, and by their ſum when they are unlike.

3. Add the quotient, laſt found, to the number belonging to the leaſt error, when that number is too little, and ſubtract it when too great, and the reſult will give the true root *nearly*.

4. Take this root and the neareſt of the former, and, by proceeding in like manner, a root will be had ſtill nearer than before; and ſo on to any degree of exactneſs required.

EXAMPLES:

1. Given $x^x = 100$, to find the value of x by approximation.

By the nature of logarithms $x \times log. x = log. 100 = 2$.

And, ſince x is found, by trial, to be greater than 3 *and leſs than* 4;

Let, therefore, 3.5 *and* 3.6 *be the two ſuppoſed values of x.*

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* The rule for ſolving exponential equations was invented by M. *Jean Bernoulli*, and publiſhed in the *Leipſic* Acts, 1697.

Then the log. x=*log.* 3.5=.5440680; *and* $x \times$ *log.* x=1.9042380

$$\begin{array}{r} 1.9042380 \\ 2 \\ \hline -.0957620 \end{array}$$

=1*st error, too little.*

And the log. x=*log.* 3.6=.5563025; *and* $x \times$ *log.* x=2.0026890

$$\begin{array}{r} 2.0026890 \\ 2 \\ \hline .0026890 \end{array}$$

=2*d error, too great.*

1*st number*	3.5	1*st error*	−.095762
2*d number*	3.6	2*d error*	+.002689
	0.1=*diff.*		.098451=*sum.*

$$\frac{0.1 \times .002689}{.098451} = .00273 = \textit{correction},$$

2*d number*	3.60000
correction	−.00273
	3.59727=x=*root nearly.*

Again, suppose x=3.597; *then we shall have* *log.* x= .5559404, *and* $x \times$ *log.* x=1.9997176, *which subtracted from* 2 *gives* .0002824 *the* 3*d error, too little.*

2*d number*	3.600	2*d error*	+.0026890
3*d number*	3.597	3*d error*	−.0002824
	.003=*diff.*		.0029714=*sum.*

$$\frac{.003 \times .0002824}{.0029714} = .000285 \textit{ the correction.}$$

3*d number*	3.597000
correction	+0.000285
	3.597285=x= *root required nearly.*

OF INDETERMINATE, OR UNLIMITED PROBLEMS.

A problem is said to be *indeterminate* or *unlimited*, when the equations, expressing the conditions of a question, are less in number than the unknown quantities to be determined. And though such kind of problems are capable of innumerable answers, yet the results, in whole numbers, are generally limited to some determinate number, and may be obtained as follows.

PROBLEM I.

To find the values of x and y, in the equation $ax = by + c$; where a, b, and c, are given numbers, which admit of no common divisor.

RULE.

1. Let *wh.* stand for a *whole number*, and reduce the equation to $x = \frac{by+c}{a} = wh.$

2. Make $\frac{by+c}{a} =$ to $\frac{dy+f}{a}$, by throwing all whole numbers out of it, till d and f be each less than a.

3. Subtract $\frac{dy+f}{a}$, or some multiple of it, from $\frac{ay}{a}$, $\frac{2ay}{a}$, $\frac{3ay}{a}$, or any other multiple of y, that comes

near the former, and the remainder will be a whole number.

4. Take this remainder, or any multiple of it, from ſome of the foregoing fractions, or from any whole number, which is nearly equal to it, and the remainder, in this caſe, will alſo be a whole number.

5. Proceed in the ſame manner with this laſt remainder; and ſo on till the co-efficient of y becomes equal to 1; or $\frac{y+g}{a}=wh.=p$.

6. Then will $y=ap-g$; where p may be any whole number whatever; and y being known, x may be found from the given equation.

EXAMPLES:

1. Given $19x=14y-11$, to find x and y in whole numbers.

$$\textit{First, } x=\frac{14y-11}{19}=wh.;\ \textit{and } \frac{19y}{19}=wh.$$

$$\textit{Then, by ſubtraction, } \frac{19y}{19}-\frac{14y-11}{12}=\frac{5y+11}{19}=wh.$$

$$\textit{Again, } \frac{5y+11}{19}\times 4=\frac{20y+44}{19}=\frac{20y+6}{19}+2=wh.$$

$$\textit{or } \frac{20y+6}{19}=wh.$$

$$\textit{Therefore } \frac{20y+6}{19}-\frac{19y}{19}=\frac{y+6}{19}=wh.=p.$$

And $y=19p-6$; *where, if* p *be taken* $=1$, *for the leaſt affirmative value of* y, *we ſhall have* $y=13$, *and* $x=9$, *the anſwer.*

2. Given $3x=8y-16$, to find the values of x and y in whole numbers.

Here, $x=\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}=$ *wh. or* $\frac{2y-1}{3}=$ *wh.*

And, $\frac{2y-1}{3}\times 2=\frac{4y-2}{3}=$ *wh. But* $\frac{3y}{3}$ *is also* $=$ *wh.*

Therefore $\frac{4y-2}{3}-\frac{3y}{3}=\frac{y-2}{3}=$ *wh.* $=p$.

Whence $y=3p+2$; *and, by taking* $p=0$, *we shall have* $y=2$ *and* $x=0$, *the answer.*

3. Given $9x+13y=2000$; to find all the possible values of x and y in whole numbers.

First, $x=\frac{2000-13y}{9}=222-y+\frac{2-4y}{9}=$ *wh. or* $\frac{2-4y}{9}=$ *wh.*

And, $\frac{2-4y}{9}\times 2=\frac{4-8y}{9}=$ *wh. But,* $\frac{9y}{9}$ *is also* $=$ *wh.*

Therefore $\frac{9y}{9}+\frac{4-8y}{9}=\frac{y+4}{9}=$ *wh.* $=p$.

Whence $y=9p-4$; *and, by taking* $p=1$, *we shall have* $y=5$ *and* $x=215$.

And, by adding 9, *continually, to the last value of* y, *and subtracting* 13 *from that of* x, *all the possible answers will stand as follows:*

$$x=\begin{cases}215.189.163.137.111.85.59.33.7\\202.176.150.124.\ 98.72.46.20\end{cases}$$

$$y=\begin{cases}5.23.41.59.77.\ 95.113.131.149\\14.32.50.68.86.104.122.140\end{cases}$$

4. Given $24x=13y+16$, to find x and y in whole numbers. *Anſ.* $x=5$, *and* $y=8$.

5. Given $14x=4y+7$, to find x and y in whole numbers. *Anſ. The queſtion is impoſſible.*

6. Given $27x=1600-16y$, to find x and y in whole numbers. *Anſ.* $y=19$, *and* $x=48$.

7. Given $87x+256y=15410$, to find the leaſt value of x, and the greateſt of y, in whole poſitive numbers. *Anſ.* $x=30$, *and* $y=12800$.

8. Given $5x+7y+11z=224$, to find all the poſſible values of x, y, and z, in whole numbers.

Anſ. The number of anſwers is 60.

9. How many different ways is it poſſible to pay 100 l. in guineas and piſtoles only; a guinea being equal to 21 s. and a piſtole to 17 s.

Anſ. 6 *different ways.*

10. To determine whether it be poſſible to pay 100 l. in guineas and moidores only.

Anſ. The queſtion is impoſſible.

11. A perſon bought 20 birds, of three different ſorts, for 1 s. 8 d. the firſt ſort at 4 d. the ſecond at $\frac{1}{2}$ d. and the third at $\frac{1}{4}$ a-piece: how many had he of each?

Anſ. 3 *of the* 1*ſt ſort,* 15 *of the* 2*d, and* 2 *of the* 3*d.*

12. I owe my friend a ſhilling, and have nothing about me but guineas, and he has nothing but louis d'ors; the queſtion is, how muſt I acquit myſelf of the debt? the louis d'ors being valued at 17 s. *Anſ. I muſt pay him* 13 *guineas, and he muſt give me* 16 *louis d'ors.*

13. To find in what year of Chriſt the cycle of the ſun was 8, the cycle of the moon 10, and the cycle of indiction 10. *Anſ. In the year* 1567.

14. Twenty perſons, conſiſting of men, women, and children, ſpend altogether 20 s. the men ſpend 4 s. a-piece; the women 6 d. and the children 3 d. how many were there of each ſort?

Anſ. 3 *men,* 15 *women, and* 2 *children.*

15. A vintner has winé at 2s. 1s. 10d. and 1s. 6d. *per* gallon: how much of each sort must he take, so as to make a mixture of 30 gallons, to be sold at 1s. 8d. *per* gallon?

Ans. 16, 2, 12; 17, 4, 9; 18, 6, 6; *or* 19, 8, 3 *of each sort.*

16. To determine how many ways it is possible to pay 1000l. in crowns, guineas, and moidores, only. *Ans.* 70734 *different ways.*

PROBLEM II.

To find such a whole number x, as being divided by the given numbers a, b, c, &c. shall leave the given remainders f, g, h, &c.

RULE.

1. Subtract each of the remainders from x, and divide the difference by a, and there will result $\frac{x-f}{a}$, $\frac{x-g}{a}$, $\frac{x-h}{a}$, &c. = whole numbers.

2. Call the value of x, in the first fraction, p, and substitute this quantity in the place of x in the second fraction.

3. Find the least value of p, in the second fraction, by the last problem, and call it r.

4. Let the value of x be found in terms of r, and substitute this quantity in the place of x in the third fraction.

5. Find the least value of r in the third fraction, and call it s; and the value of x in terms of s, being substituted for x in the fourth fraction, and so on, will give the whole number required.

EXAMPLES:

1. To find the leaſt whole number, which, by being divided by 17, ſhall leave a remainder of 7; but being divided by 26, the remainder ſhall be 13.

Let $x=$ *number required.*

$$\textit{Then } \frac{x-7}{17}, \textit{ and } \frac{x-13}{26} = \textit{whole numbers.}$$

$$\textit{And, by putting } \frac{x-7}{17}=p, \textit{ we ſhall have } x=17p+7;$$

Which value of x, *being ſubſtitued in the 2d fraction,*

$$\textit{gives } \frac{17p-6}{26}=wh. \textit{ But } \frac{26p}{26} \textit{ is alſo } =wh.$$

$$\textit{And, therefore, } \frac{26p}{26}-\frac{17p-6}{26}=\frac{9p+6}{26}=wh.$$

$$\textit{or } \frac{9p+6}{26}\times 3=\frac{27p+18}{26}=p+\frac{p+18}{26}=wh. \textit{ or } \frac{p+18}{26} =wh. =r.$$

Hence $p=26r-18$, *and, by taking* $r=1$, *we ſhall have* $p=8$.

And, conſequently $x=17\times 8+7=143$, *the number required.*

2. To find a number which being divided by 11, 19, and 29, the remaiders ſhall be 3, 5, and 10.

Let $x=$ *number required.*

$$\textit{Then } \frac{x-3}{11}, \frac{x-5}{19} \textit{ and } \frac{x-10}{29} = \textit{whole numbers.}$$

$$\textit{And, by putting } \frac{x-3}{11}=p, \textit{ we ſhall have } x=11p+3;$$

Which value of x, *being substituted in the 2d fraction, gives* $\frac{11p-2}{19}=$ *wh. or* $\frac{11p-2}{19}\times 2=\frac{22p-4}{19}=p+\frac{3p-4}{19}=$ *wh.*

Also, $\frac{3p-4}{19}=$ *wh.* $\frac{3p-4}{19}\times 6=\frac{18p-24}{19}=\frac{18p-5}{19}-1=$ *wh. or* $\frac{18p-5}{19}=$ *wh.*

But $\frac{19p}{19}$ *is, likewise,* $=$ *wh. and, therefore,* $\frac{19p}{19}-\frac{18p-5}{19}=\frac{p+5}{19}=$ *wh.* $=r$.

Hence, $p=19r-5$, *and* $x=\overline{19r-5}\times 11+3=209r-52$.

And, by substituting this value of x *in the* 3*d fraction, we shall have* $\frac{209r-62}{29}=7r-2+\frac{6r-4}{29}=$ *wh. or* $\frac{6r-4}{29}=$ *wh.*

But, $\frac{6r-4}{29}\times 5=\frac{30r-20}{29}=r+\frac{r-20}{29}=$ *wh. or* $\frac{r-20}{29}=$ *wh.* $=s$.

Therefore, $r=29s+20$; *and, by putting* $s=0$, *we shall have* $r=20$.

And, consequently, $x=209\times 20-52=4128=$ *number required.*

3. To find the least whole number, which being divided by 19, shall leave a remainder of 7; but being divided by 28, the remainder shall be 13.

Ans. 349.

4. To find a number, which being divided by 3, 5, 7, and 2; will leave the remainders 2, 4, 6, and 0, respectively. *Anſ.* 104.

5. To find the leaſt whole number, which, being divided by 16, 17, 18, 19, and 20, ſhall leave 6, 7, 8, 9, and 10 remainders. *Anſ.* 232550.

6. To find the leaſt whole number, which, being divided by the nine digits, reſpectively, ſhall leave no remainders. *Anſ.* 2520.

DIOPHANTINE PROBLEMS.

* *Diophantine problems* are thoſe which relate to the finding of ſquare and cube numbers, &c. and are ſuch as are generally capable of a great variety of anſwers. They are ſo called from their inventor *Diophantus* of *Alexandria* in *Egypt*, who flouriſhed in or about the third century, and is the firſt writer on Algebra we meet with amongſt the ancients.

Theſe queſtions are ſo exceedingly curious and abſtruſe, that nothing leſs than the moſt refined Algebra, applied with the utmoſt ſkill and judgment,

* That *Diophantus* was not the inventor of algebra, as has been generally imagined, is exceedingly obvious; for his method of applying it is ſuch, as could only have been uſed in a very advanced ſtate of the ſcience. He no where ſpeaks of the fundamental rules and principles, as an inventor certainly would have done, but treats of it as an art already ſufficiently known; and ſeems to intend, not ſo much to teach it, as to cultivate and improve it, by ſolving ſuch queſtions as, before this time, had been thought too difficult to be ſurmounted.

could ever ſurmount the difficulties which attend them. And, in this way, no man has ever extended the limits of the analytic art further than *Diophantus*, or diſcovered greater penetration and judgment in the application of it.

When we conſider his work with attention, we are at a loſs which to admire moſt, his wonderful ſagacity, and peculiar artifice, in forming ſuch poſitions as the nature of the problems required, or the more than ordinary ſubtilty of his reaſoning upon them.

Every particular queſtion puts us upon a new way of thinking, and furniſhes a freſh vein of analytical treaſure, which cannot but be very inſtructive to the mind, in conducting it through almoſt all difficulties of this kind, wherever they occur.

The following method of reſolving theſe queſtions will be found of conſiderable ſervice; but no general rule can be given, that will ſuit all caſes; and therefore the ſolution muſt often be left to the ſagacity and ſkill of the learner.

It is probable, therefore, that Algebra was known in the world, long before the time of *Diophantus*; but that the works of preceding writers have been deſtroyed by the ravages of time, or the depredations of ignorant barbarians.

His *Arithmetics*, out of which theſe problems were moſtly collected, conſiſted originally of thirteen books; but the firſt ſix only are now extant. The beſt edition is ſaid to be that publiſhed at *Paris*, by Monſieur *Bachet*, in the year 1621. And in this work, the ſubject is ſo ſkilfully handled, that the moderns, notwithſtanding their other improvements, have been able to do little more than explain and illuſtrate his method.

Thoſe who have ſucceeded beſt, in this reſpect, are *Victa*, *Brancker*, *Kerſey*, *De Billy*, *Ozanam*, *Preſtet*, *Saunderſon*, *Fermat*, and *Euler*.

RULE.

1. For the root of the square or cube required, put one or more letters such, that when they are involved, either the given number, or the highest power of the unknown quantity, may vanish from the equation; and then if the unknown quantity be but of one dimension, the problem will be solved by reducing the equation.

2. But if the unknown quantity be still a square, or a higher power, some other new letters must be assumed to denote the root; with which proceed as before; and so on till the unknown quantity be but of one dimension; and from this all the rest will be determined.

EXAMPLES:

1. *To divide a given square number (100) into two such parts, that each of them may be a square number.

Let x^2 $(=\square)$ *be one of the parts, and then* $100-x^2$ *will be the other part, which is also to be a square number.*

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* If $x-10$ had been made the side of the second square, in this question, instead of $2x-10$, the equation would have been $x^2-20x+100=100-x^2$; in which case, x, the side of the first square, would have been found $=10$, and $x-10$, or the side of the second square $=0$; and for this reason the substitution, $x-10$, was avoided; but $3x-10$, $4x-10$, or any other quantity of the same kind, would have succeeded equally as well as the former; though, in some cases, the results would have been less simple.

Assume the side of this second square $=2x-10$.
Then will $100-x^2=\overline{2x-10}|^2=4x^2-40x+100$;
And, by reduction, $x=8$, *and* $2x-10=6$.
Therefore 64 *and* 36 *are the parts required.*

THE SAME GENERALLY.

Let $a^2=$ *given square number,* $x^2\ (=\square)=$ *one of its parts, and* $a^2-x^2=$ *the other, which is also to be a square number.*

Assume the side of this second square $=rx-a$,
then will $a^2-x^2=\overline{rx-a}|^2=r^2x^2-2arx+a^2$;
And, by reduction, $x=\frac{2ar}{r^2+1}$, *and* $rx-a=\frac{2ar^2}{r^2+1}-a=\frac{ar^2-a}{r^2+1}$.

Therefore $\overline{\frac{2ar}{r^2+1}}|^2$ *and* $\overline{\frac{ar^2-a}{r^2+1}}|^2$ *are the parts required; where* a *and* r *may be any numbers, taken at pleasure.*

* 2. To divide a given number (13) consisting of two known square numbers (9 and 4) into two other square numbers.

If s and r be any two unequal numbers, of which s is the greater; then will $2rs$, s^2-r^2 and s^2+r^2, be the perpendicular, base, and hypothenuse of a right-angled triangle.

And from this canon two square numbers may be found, whose sum or difference shall be square numbers; for $\overline{2rs}|^2+\overline{s^2-r^2}|^2=\overline{s^2+r^2}|^2$, and $\overline{r^2+s^2}|^2-\overline{2rs}|^2=\overline{s^2-r^2}|^2$, or $\overline{s^2+r^2}|^2-\overline{s^2-r^2}|^2=\overline{2rs}|^2$; and this when s and r are any numbers whatever.

* This question is considered, by *Diophantus*, as a very important one, being made the foundation, or *basis*, of most

And, consequently, $y=\frac{r^2-2rs^2}{4rs+1}$ *and* $x=\frac{r^2-y}{2r}=\frac{2r^2s+s^2}{4rs+1}$.

So that $\frac{r^2-2rs^2}{4rs+1}$ *and* $\frac{2r^2s+s^2}{4rs+1}$ *are the numbers required; where r and s may be taken at pleasure, provided that r be greater than* $2s^2$.

5. To find two numbers, whose sum and difference shall be both square numbers.

Let x and x^2-x *be the two numbers sought.*

Then, since their sum is evidently a square number, one of the conditions of the questions will be answered.

There remains, therefore, only their difference x^2-2x *to be made a square.*

And, if for the side of this square, there be put $x-r$, *we shall have* $x^2-2rx+r^2=x^2-2x$, *or* $2rx-2x=r^2$.

Whence, $x=\frac{r^2}{2r-2}$ *and* $x^2-x+\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$.

So that $\frac{r^2}{2r-2}$ *and* $\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$ *are the numbers required; where r may be taken at pleasure, provided it be greater than* 1.

6. To find three numbers such, that not only the sum of all three of them, but also the sum of every two shall be a square number.

Let $4x$, x^2-4x *and* $2x+1$ *be the three numbers sought.*

Then $\overline{4x+x^2-4x}$ (x^2), $\overline{x^2-4x+2x+1}$ (x^2-2x+1), *and* $\overline{4x+x^2-4x+2x+1}$ (x^2+2x+1), *are evidently squares.*

And, therefore, three of the conditions, mentioned in the question, are accomplished.

Whence, it remains only, to make the quantity $\overline{4x}+\overline{2x+1}$, *or* $6x+1=$ *to a square.*

Let, therefore, $6x+1=a^2$; *and we shall have* $x=\frac{a^2-1}{6}$.

And, consequently, $\frac{4a^2-4}{6}$, $\overline{\frac{a^2-1}{6}}\Big|^2-\frac{4a^2-4}{6}$, *and* $\frac{2a^2-2}{6}+1$; *or* $\frac{2a^2-2}{3}$, $\frac{a^4-26a^2+25}{36}$, *and* $\frac{a^2+2}{3}$ *are the numbers required; where a may be taken at pleasure, provided it be greater than* 5.

7. To find three square numbers such, that the sum of every two of them shall be a square number *.

Let x^2, y^2, *and* z^2 *be the numbers sought;*
Then $x^2+z^2=\square$, $y^2+z^2=\square$, *and* $x^2+y^2=\square$.
Or $\frac{x^2}{z^2}+1=\square$, $\frac{y^2}{z^2}+1=\square$, *and* $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\square$.

And, by putting $\frac{x}{z}=\frac{s^2-1}{2s}$, *and* $\frac{y}{z}=\frac{r^2-1}{2r}$,

we shall have $\frac{x^2}{z^2}+1=\frac{s^4+2s^2+1}{4s^2}$, *and* $\frac{y^2}{z^2}+1=\frac{r^4+2r^2+1}{4r^2}$ *which are both evidently squares; and*

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* This question is capable of a great variety of answers; but the least roots, which have yet been found, in whole numbers, are 44, 117, and 240. See *Elémens d'Algèbre, par M. Euler,* tome II. page 327.

therefore it remains only to make $\frac{x^2}{z^2}+\frac{y^2}{z^2}=$ *square number.*

But $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\overline{\frac{s^2-1}{2s}}\Big|^2+\overline{\frac{r^2-1}{2r}}\Big|^2=\frac{\overline{s^2-1}|^2}{4s^2}+\frac{\overline{r^2-1}|^2}{4r^2}$

$=\frac{4r^2\times\overline{s^2-1}|^2+4s^2\times\overline{r^2-1}|^2}{4r^2s^2}=\square$;

Or $r^2\times\overline{s^2-1}|^2+s^2\times\overline{r^2-1}|^2=r^2\times\overline{s+1}|^2\times\overline{s-1}|^2+s^2\times\overline{r+1}|^2\times\overline{r-1}|^2=\square$.

And, by making $r-1=s+1$, *or* $r=s+2$, *we shall have* $\overline{s+2}|^2\times\overline{s+1}|^2\times\overline{s-1}|^2+s^2\times\overline{s+3}|^2\times\overline{s+1}|^2=\square$;

Or $\overline{s+2}|^2\times\overline{s-1}|^2+s^2\times\overline{s+3}|^2=2s^4+8s^3+6s^2-4s+4=\square$.

Now, let the root of this square be assumed $=\frac{5}{4}s^2-s+2$,

Then, $2s^4+8s^3+6s^2-4s+4=\overline{\frac{5}{4}s^2-s+2}|^2=\frac{25}{16}s^4-\frac{5}{2}s^3+5s^2+s^2-4s+4$; *or* $2s^4+8s^3=\frac{25}{16}s^4-\frac{5}{2}s^3$; *or* $2s+8=\frac{25}{16}s-\frac{5}{2}$.

Whence $s=-24$, *and* $r=-22$.

And, consequently, $\frac{x}{z}=\frac{s^2-1}{2s}=-\frac{575}{48}$, *and* $\frac{y}{z}=\frac{r^2-1}{2r}=-\frac{483}{44}$;

or $x=-\frac{575z}{48}$, *and* $y=-\frac{483z}{44}$.

In order, therefore, to have the answer in whole numbers, let $z=528$, *and we shall have* $x=6325$, *and* $y=5796$, *or* 528, 5796 *and* 6325 *for the roots of the squares required.*

8. To find a number x such, that $x+1$ and $x-1$ shall be both square numbers. *Ans.* $x=\frac{5}{4}$.

9. To find a number x such, that $x+128$ and $x+192$ shall be both squares. *Ans.* $x=97$.

10. To find a number x such, that x^2+x and x^2-x may be both squares. *Ans.* $x=\frac{25}{24}$.

11. To find two numbers x and y such, that $x+y$, x^2+y *and* y^2+x may be all squares.

Ans. $x=\frac{1}{6}$ *and* $y=\frac{1}{12}$.

12. To find three numbers in arithmetical progression such, that the sum of every two of them may be a square number. *Ans.* $120\frac{1}{2}$, $840\frac{1}{2}$, *and* $1560\frac{1}{2}$.

13. To find three numbers such, that if to the square of every one of them the sum of the other two be added, the three sums shall be all squares.

Ans. $\frac{8}{3}$, $\frac{16}{3}$ *and* 1.

14. To find two numbers in proportion as 8 is to 15, and such that the sum of their squares shall make a square number. *Ans.* 576 *and* 1080.

15. To find three square numbers in arithmetical progression. *Ans.* 1, 25, *and* 49.

16. To find three square numbers in harmonical proportion. *Ans.* 1225, 49, *and* 25.

17. To find four numbers such, that if a square number (100) be added to the product of every two of them, the sums shall be all squares.

Ans. 2, 32, 88 *and* 168.

18. To find two numbers such, that their difference may be equal to the difference of their squares, and that the sum of their squares shall be a square number. *Ans.* $\frac{4}{7}$ *and* $\frac{3}{7}$.

19. To find three numbers in geometrical proportion such, that every one of them being increased by a given number 19, shall make square numbers.

Ans. 81, $\frac{5}{4}$ *and* $\frac{25}{1296}$.

20. To find two numbers such, that if their product be added to the sum of their squares, it shall make a square number.

Ans. 5 *and* 3, 8 *and* 7, 16 *and* 5, &c.

21. To divide a given number (10) into four such parts, that the sum of every three of them may make a square number. *Ans.* $1, 6, \frac{186}{289}$ *and* $\frac{681}{289}$.

22. To find three square numbers such, that their sum being severally added to their three sides shall make square numbers.

Ans. $\frac{4418}{62920}$, $\frac{13254}{62920}$, *and* $\frac{19881}{62920}$ = *roots required.*

23. To find two numbers such, that their sum being increased and lessened, either by their difference, or the difference of their squares, the sums and remainders shall be all squares.

Ans. $\frac{49}{50}$ *and* $\frac{1}{50}$.

24. To find two numbers such, that not only each number, but also their sum and their difference, being increased by unity, shall be all square numbers. *Ans.* 3024 *and* 5624.

25. To find three numbers such, that whether their sum be added to, or subtracted from the square of every particular number, the numbers thence arising shall be all squares.

Ans. $\frac{406}{96}$, $\frac{518}{96}$ *and* $\frac{791}{96}$.

26. To find three square numbers such, that the sum of their squares shall also be a square number.

Ans. 9, 16, *and* $\frac{144}{25}$.

27. To find three square numbers such, that the difference of every two of them shall be a square number. *Ans.* 485809, 34225 *and* 23409.

28. To divide any given cube number (8) into three other cube numbers. *Ans.* $\frac{64}{27}$, $\frac{125}{27}$ *and* 1.

29. Two cube numbers (8 and 1) being given, to find two other cube numbers, whose difference shall be equal to the sum of the given cubes.

Anſ. $\frac{8000}{343}$, *and* $\frac{4913}{343}$.

30. To divide a given number (28) composed of two cube numbers (27 and 1) into two other cube numbers. *Anſ.* $\frac{63284705}{21446828}$ *and* $\frac{28340511}{21446828}$ *the roots.*

31. To find three cube numbers such, that if from every one of them a given number (1) be subtracted, the sum of the remainders shall be a square.

Anſ. $\frac{4913}{3375}$, $\frac{21952}{3375}$ *and* 8.

32. To find three numbers such, that if they be severally added to the cube of their sum, the three sums thence arising shall be all cubes.

Anſ. $\frac{1538}{157464}$, $\frac{18577}{157464}$ *and* $\frac{23625}{157464}$.

33. To find three numbers in arithmetical proportion such, that the sum of their cubes shall be a cube. *Anſ.* 3, 4, 5, *or* 149, 256, *and* 363, *&c.*

34. To find three cube numbers such, that their sum shall be a cube number.

Anſ. 3^3, 4^3, *and* 5^3, *or* 21^3, 19^3, 18^3, *&c.*

Of the SUMMATION, and INTERPOLATION, of INFINITE CONVERGING SERIES.

The *doctrine of infinite series* is a subject which has engaged the attention of the greatest Mathematicians in all ages; and is, perhaps, one of the most abstruse and difficult branches of abstract mathematics.

To find the ſum of a ſeries, the number of whoſe terms is inexhauſtible, or infinite, has been conſidered by ſome as an extravagant paradox, or a thing impoſſible to be done. But this difficulty will be eaſily removed, by conſidering, that every finite magnitude whatever is diviſible *in infinitum*, or conſiſts of an infinite number of parts, whoſe aggregate, or ſum, is equal to the quantity firſt propoſed.

A number actually infinite is, indeed, a plain contradiction to all our ideas; for any number which we can poſſibly conceive, or of which we have any notion, muſt always be determinate and finite; ſo that a greater may be ſtill aſſigned, and a greater after this; and ſo on, without a poſſibility of ever coming to an end of the increaſe or addition.

And this inexhauſtibility, in the nature of numbers, is, therefore, all that we can diſtinctly comprehend by their infinity; for though we can eaſily conceive that a finite quantity may become greater and greater without end, yet we are not from thence enabled to form any notion of the *ultimatum*, or laſt magnitude, which is incapable of further augmentation.

We cannot, therefore, apply to an infinite ſeries the common notion of a ſum, or a collection of ſeveral particular numbers, that are joined and added together, one after another; for this ſuppoſes, that thoſe particulars are all known and determined. But as every ſeries generally obſerves ſome particular law, and continually approaches towards a term or limit, we can eaſily conceive it to be a whole, of its own kind, and that it muſt have a certain real value, whether that value be determinable or not.

Thus, in many ſeries, a number is aſſignable, beyond which, no number of its terms can ever

reach, or indeed ever be equal to it; but yet may approach to it, in such a manner, as to want less than any given difference. And this we may call the value, or sum of the series; not as being a number found by the common method of addition, but as such a limitation of the value of the series, taken in all its infinite capacity, that if it were possible to add them all together, one after another, the sum would be equal to that number.

Again, in other series, the value has no limitation; and this may be expressed by saying, that the sum of the series is infinitely great; or, which is the same thing, that it has no determinate and assignable value; but may be carried on, to such a length, as that its sum shall exceed any given number whatever.

According to the common rule for summing up a finite progression of a geometric decreasing series, where r is the ratio, l the first term, and a the least, the sum is $\frac{rl-a}{r-1}$. And if we suppose a, the less extreme, to be actually decreased to o, then the sum of the whole series will be $\frac{rl}{r-1}$. For it is demonstrable, that the sum of no assignable number of terms of the series can ever be equal to that quotient; and yet no number less than it, will ever be equal to the value of the series.

Whatever consequences, therefore, follow from the supposition of $\frac{rl}{r-1}$ being the true and adequate value of the series, taken in all its infinite capacity, as if all the parts were actually determined and added together, they can never be the occasion of any assignable error, in any operation or demonstration,

where it is used in that sense; because if you say that it exceeds that value, it is demonstrable that this excess must be less than any assignable difference, which is, in effect, no difference at all; and therefore, the supposed error will likewise be no error, and consequently $\frac{rl}{r-1}$ may be looked upon as expressing the adequate and just value of the infinite series.

But we are further satisfied of the reasonableness of this doctrine, by finding, in fact, that a finite quantity does actually convert into an infinite series, as appears in the case of circulating decimals. Thus, $\frac{2}{3}$ turned into a decimal is $=.6666$, &c. $=\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}$, &c. continued *ad infinitum*. But this is plainly a geometric series, from $\frac{6}{10}$, in the continued ratio of 10 to 1, and the sum of all its terms, continued to infinity, will evidently be equal to $\frac{2}{3}$, or the number from whence it was originally derived. And the same may be shewn of many other series, and of all circulating decimals in general.

PROBLEM I.

Any series being given to find the several orders of differences.

RULE.

1. Take the first term from the second, the second from the third, the third from the fourth, &c. and the remainders will form a new series, called the *first order of differences.*

2. Take the first term of this last series from the second, the second from the third, the third from

the fourth, &c. and the remainders will form another new series, called the *second order of differences.*

3. Proceed, in like manner, for the *third*, *fourth*, *fifth*, &c. orders of differences; and so on till they terminate, or are carried as far as is thought necessary.

EXAMPLES:

1. To find the several orders of differences in the series 1, 4, 9, 16, 25, 36, &c.

1,	4,	9,	16,	25,	36,	&c.
	3,	5,	7,	9,	11,	&c. *1st diff.*
		2,	2,	2,	2,	&c. *2d diff.*
			0,	0,	0,	&c.

2. To find the several orders of differences in the series 1, 8, 27, 64, 125, 216, &c.

1,	8,	27,	64,	125,	216,	&c.
	7,	19,	37,	61,	91,	&c. *1st diff.*
		12,	18,	24,	30,	&c. *2d diff.*
			6,	6,	6,	&c. *3d diff.*
				0,	0,	&c.

3. To find the several orders of differences in the series 1, 3, 6, 10, 15, 21, &c.

Ans. *1st* 2, 3, 4, 5, *&c.* *2d* 1, 1, 1, *&c.*

4. To find the several orders of differences in the series 1, 6, 20, 50, 105, 196, &c.

Ans. *1st* 5, 14, 30, 55, 91, *&c.* *2d* 9, 16, 25, 36, *&c.* *3d* 7, 9, 11, *&c.* *4th* 2, 2, *&c.*

PROBLEM II.

Any series a, b, c, d, e, &c. being given, to find the first term of the nth. order of differences.

RULE.

Let δ stand for the first term of the nth. differences.

Then will $a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}e$, &c. to $n+1$ terms $=\delta$, when n is an even number.

And $-a+nb-n\times\frac{n-1}{2}c+n\times\frac{n-1}{2}\times\frac{n-2}{3}d-n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}e$, &c. to $n+1$ terms $=\delta$, when n is an odd number.

EXAMPLES:

1. Required the first term of the third order of differences, of the series 1, 5, 15, 35, 70, &c.

Let $a, b, c, d, e,$ *&c.* $=1, 5, 15, 35, 70,$ *&c. and* $n=3$.

Then $-a+nb-n\times\frac{n-1}{2}c+n\times\frac{n-1}{2}\times\frac{n-2}{3}d=-a+3b-3c+d=-1+15-45+35=4=$ *the first term required.*

2. Required the first term of the fourth order of differences of the series, 1, 8, 27, 64, 125, &c.

Let $a, b, c, d, e,$ *&c.* $=1, 8, 27, 64, 125,$ *&c. and* $n=4$.

Then $a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d+n\times$

$\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} e$, $= a - 4b + 6c - 4d + e = 1 - 32 + 162 - 256 + 125 = 0$; *so that the first term of the fourth order is* 0.

3. Required the first term of the fifth order of differences, of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c.

Ans. $-\frac{1}{32}$.

4. Required the first term of the 8th order of differences of the series, 1, 3, 9, 27, 81, &c. *Ans.* 256.

PROBLEM III.

To find the nth. term of the series, a, b, c, d, e, &c.

RULE.

Let d^{I}, d^{II}, d^{III}, d^{IV}, &c. be the first of the several orders of differences, found as in the last problem:

Then will $a + \frac{n-1}{1} d^{I} + \frac{n-1}{1} \times \frac{n-2}{2} d^{II} + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} d^{III} + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} d^{IV}$, &c. be = nth. term required.

EXAMPLES.

1. To find the 12th term of the series 2, 6, 12, 20, 30, &c.

2, 6, 12, 20, 30, &c.
4, 6, 8, 10, &c.
2, 2, 2, &c.
0, 0, &c.

Here 4 and 2 are the first terms of the differences;
Let, therefore, $4=d^{I}$, $2=d^{II}$, and $n=12$.

Then $a+\frac{n-1}{1}d^{I}+\frac{n-1}{1}\times\frac{n-2}{2}d^{II}=2+11d^{I}+55d^{II}=2+44+110=156=$ *12th term required.*

2. Required the 20th term of the series 1, 3, 6, 10, 15, 21, &c.

1, 3, 6, 10, 15, 21, &c.
2, 3, 4, 5, 6, &c.
1, 1, 1, 1, &c.
0, 0, 0, &c.

Here 2 and 1 are the first terms of the differences.
Let, therefore, $2=d^{I}$, $1=d^{II}$, and $n=20$,

Then $a+\frac{n-1}{1}d^{I}+\frac{n-1}{1}\times\frac{n-2}{2}d^{II}=1+19d^{I}+171d^{II}=1+38+171=210=$ *20th term required.*

3. Required the 15th term of the series 1, 4, 9, 16, 25, 36, &c. *Ans.* 225.

4. Required the 20th term of the series 1, 8, 27, 64, 125, &c. *Ans.* 8000.

PROBLEM IV.

To find the sum of n terms of the series a, b, c, d, e, &c.

RULE.

Let d^{I}, d^{II}, d^{III}, d^{IV}, &c. be the first of the several orders of differences.

Then will $na+n\times\frac{n-1}{2}d^{I}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{II}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d^{III}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times$

$\frac{n-3}{4}\times\frac{n-4}{5}d^{iv}$, &c. $=$ to the sum of n terms of the series.

EXAMPLES:

1. To find the sum of n terms of the series 1, 2, 3, 4, 5, 6, &c.

1, 2, 3, 4, 5, 6, &c.
1, 1, 1, 1, 1, &c.
0, 0, 0, 0, &c.

Here 1 *and* 0 *are the first terms of the differences.*
Let, therefore, $a=1$, $d^{i}=1$, *and* $d^{ii}=0$;
Then will $na+n\times\frac{n-1}{2}d^{i}=n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=$ *sum of* n *terms, as required.*

2. To find the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, 5^2, &c. or 1, 4, 9, 16, 25, &c.

1, 4, 9, 16, 25, &c.
3, 5, 7, 9, &c.
2, 2, 2, &c.
0, 0, &c.

Here 3 *and* 2 *are the first terms of the differences:*
Let, therefore, $a=1$, $d^{i}=3$ *and* $d^{ii}=2$.
Then will $na+n\times\frac{n-1}{2}d^{i}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{ii}=n+3n\times\frac{n-1}{2}+n\times\frac{n-1}{2}\times\frac{n-2}{3}=\frac{3n^2-n}{2}+\frac{n^3-3n^2+2n}{3}=\frac{n\times\overline{n+1}\times\overline{2n+1}}{6}=$ *sum of* n *terms, as required.*

* 3. To find the sum of n terms of the series 1^3, 2^3, 3^3, 4^3, 5^3, &c. or 1, 8, 27, 64, 125, &c.

$$\begin{array}{ccccc} 1, & 8, & 27, & 64, & 125, \text{ \&c.} \\ & 7, & 19, & 37, & 61, \text{ \&c.} \\ & & 12, & 18, & 24, \text{ \&c.} \\ & & & 6, & 6, \text{ \&c.} \\ & & & & 0, \text{ \&c.} \end{array}$$

Here the first terms of the differences are 7, 12 *and* 6. *Let, therefore,* $a=1$, $d^{I}=7$, $d^{II}=12$, *and* $d^{III}=6$.

Then will $na+n\times\frac{n-1}{2}d^{I}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{II}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d^{III}=n+7n\times\frac{n-1}{2}+12n\times\frac{n-1}{2}\times\frac{n-2}{3}+6n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}=\frac{7n^2-5n}{2}$

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* The sums of a series of powers of the natural numbers 1, 2, 3, 4, 5, &c. may be exhibited as follows:

1	$1+2+3+4$, &c. $+n=\frac{n^2+n}{2}$
2	$1^2+2^2+3^2+4^2$, &c. $+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$
3	$1^3+2^3+3^3+4^3$; &c. $+n^3=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$
4	$1^4+2^4+3^4+4^4$; &c. $+n^4=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$
5	$1^5+2^5+3^5+4^5$, &c. $+n^5=\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}$
6	$1^6+2^6+3^6+4^6$, &c. $+n^6=\frac{n^7}{7}+\frac{n^6}{2}+\frac{n^5}{2}-\frac{n^3}{6}+\frac{n}{42}$
7	$1^r+2^r+3^r+4^r$, &c. $+n^r=\frac{n^{r+1}}{r+1}+\frac{n^r}{2}+\frac{rn^{r-1}}{3.4}-$

$\frac{r.r-1.r-2}{2.3.4.5.6}n^{r-3}+\frac{r.r-1.r-2.r-3.r-4}{2.3.4.5.6.7.6}n^{r-5}-$&c.

$$+2n^3-6n^2+4n+\frac{n^4-6n^3+11n^2-6n}{4}=\frac{n^4+2n^3+n^2}{4}$$

$=$ *sum of n terms, as required.*

4. To find the sum of n terms of the series 2, 6, 12, 20, 30, &c. *Ans.* $\frac{n\times\overline{n+1}\times\overline{n+2}}{3}$.

5. To find the sum of n terms of the series 1, 3, 6, 10, 15, &c. *Ans.* $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$.

6. To find the sum of n terms of the series 1, 4, 10, 20, 35, &c. *Ans.* $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$.

7. To find the sum of n terms of the series 1^4, 2^4, 3^4, 4^4, &c. or 1, 16, 81, 256, &c.

Ans. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$

The sums of n terms of a series of triangular numbers may also be exhibited as follows:

1	$1+1+1+1$, &c. $=n$
2	$1+2+3+4$, &c. $=\frac{n.\overline{n+1}}{1.2}$
3	$1+3+6+10$, &c. $=\frac{n.\overline{n+1}.\overline{n+2}}{1.2.3}$
4	$1+4+10+20$, &c. $=\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}}{1.2.3.4}$
5	$1+5+15+35$, &c. $=\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}.\overline{n+4}}{1.2.3.4.5}$
6	$1+6+21+56$, &c. $=\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}.\overline{n+4}.\overline{n+5}}{1.2.3.4.5.6}$
	&c. &c.

PROBLEM V.

The series a, b, c, d, e, &c. being given, whose terms are at an unit's distance from each other; to find any intermediate term by interpolation.

RULE.

Let x be the distance of any term y to be interpolated, and d^{I}, d^{II}, d^{III}, &c. the first terms of the differences:

Then will $a+xd^{I}+x\times\frac{x-1}{2}d^{II}+x\times\frac{x-1}{2}\times\frac{x-2}{3}d^{III}+x\times\frac{x-1}{2}\times\frac{x-2}{3}\times\frac{x-3}{4}d^{IV}$ &c. $=y$.

EXAMPLES.

1. Given the logarithmic sines of 1° 0′, 1° 1′, 1° 2′, and 1° 3′ to find the sine of 1° 1′ 40″.

	1° 0′	1° 1′	1° 2′	1° 3′
Sines	8.2418553	8.2490332	8.2560943	8.2630424
		71779	70611	69481
			−1168	−1130
				38

Here the first terms of the differences are 71779, −1168, *and* 38.

Let, therefore, $x=1°\ 1'\ 40''-1°\ 0'=1'\ 40''=1\frac{2}{3}$ $=$ *distance of* y, *the term to be interpolated; and* $d^{I}=$ 71779, $d^{II}=-1168$, *and* $d^{III}=38$.

Then will $y=a+xd^{I}+x\times\frac{x-1}{2}d^{II}+x\times\frac{x-1}{2}\times\frac{x-2}{3}d^{III}=a+\frac{5}{3}d^{I}+\frac{5}{9}d^{II}-\frac{5}{81}d^{III}=8.2418553+$

.0119631—.0000694—.0000002=8.2537533= *fine of* 1° 1′ 40″, *as was required.*

2. Given the series $\frac{1}{50}$, $\frac{1}{51}$, $\frac{1}{52}$, $\frac{1}{53}$, $\frac{1}{54}$, &c. to find the term which stands in the middle between $\frac{1}{52}$ and $\frac{1}{53}$. *Ans.* $\frac{1}{52.5}$.

3. Given the natural tangents of 88°, 54′, 88°, 55′, 88°, 56′, 88°, 57′, 88°, 58′, and 88°, 59′; to find the tangent of 88°, 58′, 18″. *Ans.* 55.711144.

PROBLEM VI.

Having given a series of equidistant terms a, b, c, d, e, &c. whose first differences are small; to find any intermediate term by interpolation.

RULE.

Find the value of the unknown quantity in the equation which stands against the given number of terms, in the following table, and it will give the term required.

1	$a-b=0$
2	$a-2b+c=0$
3	$a-3b+3c-d=0$
4	$a-4b+6c-4d+e=0$
5	$a-5b+10c-10d+5e-f=0$
6	$a-6b+15c-20d+15e-6f+g=0$
7	$a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d$ &c. $=0$.

EXAMPLES:

1. Given the logarithms of 101, 102, 104, and 105; to find the logarithm of 103.

Here the number of terms are 4.

Therefore against 4, in the table, we have $a-4b+6c-4d+e=0$; *or* $c=\frac{4\times\overline{b+d}-a+e}{6}$.

Whence $\begin{cases} a= 2.0043214 \\ b= 2.0086002 \\ d= 2.0170333 \\ e= 2.0211893 \end{cases}$

$4\times\overline{b+d}=16.1025340$
$a+e= 4.0255107$

$6)12.0770233$

$2.0128372=$*log. of* 103, *as required.*

2. Given the cube roots of 45, 46, 47, 48, and 49; to find the cube root of 50. *Ans.* 3.684033.

3 Given the logarithms of 50, 51, 52, 54, 55, and 56; to find the logarithm of 53.
Ans. 1.7242758695.

PROMISCUOUS EXAMPLES RELATING TO SERIES.

1. To find the ſum (S) of n terms of the ſeries 1, 2, 3, 4, 5, 6, &c.

Firſt $1+2+3+4+5$, *&c.* . . . . $n=S$.

And $n+\overline{n-1}+\overline{n-2}+\overline{n-3}+\overline{n-4}$, *&c.* . . . . $1=S$.

Therefore $\overline{n+1}+\overline{n+1}+\overline{n+1}+\overline{n+1}$, *&c.* . . . . $\overline{n+1}=2S$.

And conſequently $\overline{n+1}\times n=2S$; *or* $S=\frac{n^2+n}{2}=$ *ſum required.*

To find the sum (S) of n terms of the series 1, 3, 5, 7, 9, 11, &c.

First $1+3+5+7+9$, *&c.* .... $\overline{2n-1}=S$.

And $\overline{2n-1}+\overline{2n-3}+\overline{2n-5}+\overline{2n-7}+\overline{2n-9}$, *&c.* .... $1=S$.

Therefore $2n+2n+2n+2n+2n$, *&c.* .... $2n=2S$.

And, consequently, $2n\times n=2S$; *or* $S=\frac{2n\times n}{2}=n^2=$ *sum required.*

3. Required the sum (S) of n terms of the series $a+\overline{a+d}+\overline{a+2d}+\overline{a+3d}+\overline{a+4d}$, *&c.*

First $a+\overline{a+d}+\overline{a+2d}+\overline{a+3d}$, *&c.* ... $a+\overline{n-1}\times d=S$.

And $\overline{a+nd-d}+\overline{a+nd-2d}+\overline{a+nd-3d}+\overline{a+nd-4d}$, *&c.* .... $a=S$.

Therefore $\overline{2a+nd-d}+\overline{2a+nd-d}+\overline{2a+nd-d}$, *&c.* $\overline{2a+nd-d}=2S$.

And, consequently, $\overline{2a+nd-d}\times n=2S$; *or* $S=\overline{2a+nd-d}\times\frac{n}{2}=$ *sum required.*

OR THUS:

First, $a+\overline{a+d}+\overline{a+2d}+\overline{a+3d}+\overline{a+4d}$, *&c.*

$$=\left\{\begin{array}{l} \overline{1+1+1+1+1, \&c.}\times a \\ +\overline{0+1+2+3+4, \&c.}\times d \end{array}\right\}=S.$$

But n terms of $1+1+1+1+1$, *&c.* $=n$.

Ditto of $0+1+2+3+4$, *&c.* $=\frac{n\times\overline{n-1}}{2}$

And therefore $S=na+\frac{n\times\overline{n-1}\times d}{2}=\overline{2a+nd-d}\times\frac{n}{2}$ *as before.*

4. To find the ſum (S) of n terms of the ſeries 1, x, x^2, x^3, x^4, &c.

$$\textit{Firſt, } 1+x+x^2+x^3+x^4, \textit{ \&c.} \ldots x^{n-1}=S.$$
$$\textit{And } x+x^2+x^3+x^4+x^5, \textit{ \&c.} \ldots x^n=Sx.$$
$$\textit{Therefore } -1+x^n=Sx-S\,;$$
$$\textit{Or } S=\frac{x^n-1}{x-1}=\textit{ſum required.}$$

And, when x is a proper fraction, the ſum of the ſeries, continued ad infinitum, may be found in the ſame manner.

$$\textit{Thus, } 1+x+x^2+x^3+x^4, \textit{ \&c.}=S.$$
$$\textit{And } x+x^2+x^3+x^4+x^5, \textit{ \&c.}=Sx.$$
$$\textit{Therefore } -1=Sx-S\,; \textit{ or } S-Sx=1.$$

$\textit{Whence } S=\dfrac{1}{1-x}=$ *ſum of an infinite number of terms, as required.*

5. Required the ſum (S) of the circulating decimal .999999, &c. continued *ad infinitum.*

$$\textit{Firſt, } .999999, \textit{ \&c.}=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000} \textit{ \&c.}$$
$$=9\times:\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}, \textit{ \&c.}=S.$$
$$\textit{Or, } \frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}, \textit{ \&c.}=\frac{S}{9}.$$
$$\textit{Therefore, } 1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}, \textit{ \&c.}=\frac{10S}{9}.$$
$$\textit{And, } 1=\frac{10S}{9}-\frac{S}{9}=\frac{9S}{9}.$$
$$\textit{Whence } S=1=\textit{ſum required.}$$

6. Required the ſum (S) of the ſeries $a^2+\overline{a+d}|^2+\overline{a+2d}|^2+\overline{a+3d}|^2+\overline{a+4d}|^2$, &c. continued to n terms.

First, $a^2 = a^2$

$\overline{a+d}|^2 = a^2 + 2 \times 1ad + 1d^2$

$\overline{a+2d}|^2 = a^2 + 2 \times 2ad + 4d^2$

$\overline{a+3d}|^2 = a^2 + 2 \times 3ad + 9d^2$

$\overline{a+4d}|^2 = a^2 + 2 \times 4ad + 16d^2$

&c. &c.

$$\text{Therefore, } S = \begin{cases} \text{Sum of } n \text{ terms of } \overline{1+1+1+1, \text{ \&c.}} \times a^2 \\ + \ldots \text{ ditto of } \overline{0+1+2+3, \text{ \&c.}} \times 2ad \\ + \ldots \text{ ditto of } \overline{0+1+4+9, \text{ \&c.}} \times d^2 \end{cases}$$

But n terms of $1+1+1+1$, &c. $= n$

Ditto of $0+1+2+3$, &c. $= \dfrac{n \times \overline{n-1}}{1 \times 2}$

And ditto of $0+1+4+9$, &c. $= \dfrac{n \times \overline{n-1} \times \overline{2n-1}}{1 \times 2 \times 3}$

Whence $S = n \times a^2 + n \times \overline{n-1} \times ad + \dfrac{n \times \overline{n-1} \times \overline{2n-1}}{1 \times 2 \times 3} \times d^2 = na^2 + nad \times \overline{n-1} + \dfrac{nd^2 \times \overline{n-1} \times \overline{2n-1}}{1 \times 2 \times 3} =$ *sum required.*

7. Required the sum (S) of the series $a^3 + \overline{a+d}|^3 + \overline{a+2d}|^3 + \overline{a+3d}|^3 + \overline{a+4d}|^3$, &c. continued to n terms.

First, $a^3 = a^3$

$\overline{a+d}|^3 = a^3 + 3 \times 1a^2d + 3 \times 1ad^2 + 1d^3$

$\overline{a+2d}|^3 = a^3 + 3 \times 2a^2d + 3 \times 4ad^2 + 8d^3$

$\overline{a+3d}|^3 = a^3 + 3 \times 3a^2d + 3 \times 9ad^2 + 27d^3$

$\overline{a+4d}|^3 = a^3 + 3 \times 4a^2d + 3 \times 16ad^2 + 64d^3$

$$Therefore,\ S=\begin{cases} \text{Sum of } n \text{ terms of } \overline{1+1+1+1, \&c.} \\ \quad \times a^3 \\ + \ldots \text{ ditto of } \overline{0+1+2+3, \&c.} \\ \quad \times 3a^2d \\ + \ldots \text{ ditto of } \overline{0+1+4+9, \&c.} \\ \quad \times 3ad^2 \\ + \ldots \text{ ditto of } \overline{0+1+8+27, \&c.} \\ \quad \times d^3 \end{cases}$$

But n *terms of* $1+1+1+1$, &c. $=n$

Ditto . . . of $0+1+2+3$, &c. $=\frac{n\times\overline{n-1}}{1\times 2}$

Ditto . . . of $0+1+4+9$, &c. $=\frac{n\times\overline{n-1}\times\overline{2n-1}}{1\times 2\times 3}$

Ditto . . . of $0+1+8+27$, &c. $=\frac{n^4-2n^3+n^2}{2\times 2}$

Whence $S=na^3+\frac{n\times\overline{n-1}\times 3a^2d}{1\times 2}+$ $\frac{n\times\overline{n-1}\times\overline{2n-1}\times 3ad^2}{1\times 2\times 3}+\frac{\overline{n^4-2n^3+n^2}\times d^3}{2\times 2}=$ *sum required.*

8. Required the sum (S) of n terms of the series $1+3+7+15+31$, &c.

The terms of this series are evidently equal to 1, $\overline{1+2}$, $\overline{1+2+4}$, $\overline{1+2+4+8}$, *&c. or the successive sums of the geometrical progression* 1, 2, 4, 8, 16, *&c.*

Let, therefore, $a=1$, *and* $r=2$, *and we shall have* $a+ar+ar^2+ar^3+ar^4$, &c. $=1+2+4+8+16$, &c.

But the sums of 1, 2, 3, 4, *&c. terms of this series are*

$$\left|\begin{array}{l|l} 1 & \dfrac{ar-a}{r-1}=\overline{r-1}\times\dfrac{a}{r-1} \\ 2 & \dfrac{ar^2-a}{r-1}=\overline{r^2-1}\times\dfrac{a}{r-1} \\ 3 & \dfrac{ar^3-a}{r-1}=\overline{r^3-1}\times\dfrac{a}{r-1} \\ 4 & \dfrac{ar^4-a}{r-1}=\overline{r^4-1}\times\dfrac{a}{r-1} \\ & \text{\&c.} \qquad\qquad \text{\&c.} \end{array}\right.$$

Therefore, $S=\dfrac{a}{r-1}\times\left\{\begin{array}{l} n \textit{ terms of } r+r^2+r^3+r^4, \text{\&c.} \\ -n \textit{ terms of } 1+1+1+1, \text{\&c.} \end{array}\right.$

But $1+1+1+1$, &c. $=n$.

And $r+r^2+r^3+r^4$, &c. $=\overline{r^n-1}\times\dfrac{r}{r-1}$.

Whence $S=\overline{\overline{r^n-1}\times\dfrac{r}{r-1}-n}\times\dfrac{a}{r-1}=$ *sum required.*

9. Required the sum (S) of n terms of the series $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\frac{31}{16}$, &c. the terms of which are the successive sums of the geometrical progression $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c.

Let $a=1$ *and* $r=2$, *then will* $a+\dfrac{a}{r}+\dfrac{a}{r^2}+\dfrac{a}{r^3}+\dfrac{a}{r^4}$, &c. $=\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}$, &c.

But the sums of 1, 2, 3, 4, &c. *terms of this series are*

$$\begin{array}{l|l}
1 & \dfrac{\overline{r-1}\times a}{\overline{r-1}\times 1}=\overline{r-1}\times\dfrac{a}{r-1} \\
2 & \dfrac{\overline{r^2-1}\times a}{\overline{r-1}\times r}=\overline{r-\dfrac{1}{r}}\times\dfrac{a}{r-1}; \\
3 & \dfrac{\overline{r^3-1}\times a}{\overline{r-1}\times r^2}=\overline{r-\dfrac{1}{r^2}}\times\dfrac{a}{r-1}. \\
4 & \dfrac{\overline{r^4-1}\times a}{\overline{r-1}\times r^3}=\overline{r-\dfrac{1}{r^3}}\times\dfrac{a}{r-1}
\end{array}$$

&c. &c.

Therefore $S=\dfrac{a}{r-1}\times\left\{\begin{array}{l} n \textit{ terms of } r+r+r+r, \text{ \&c.} \\ -n \textit{ terms of } \dfrac{1}{1}+\dfrac{1}{r}+\dfrac{1}{r^2}+\dfrac{1}{r^3}, \text{ \&c.}\end{array}\right.$

But $r+r+r+r$, &c. $=nr$,

And $\dfrac{1}{1}+\dfrac{1}{r}+\dfrac{1}{r^2}+\dfrac{1}{r^3}$ &c. $=\dfrac{r^n-1}{\overline{r-1}\times r^{n-1}}$,

Whence $S=\dfrac{a}{r-1}\times nr-\dfrac{r^n-1}{\overline{r-1}\times r^{n-1}}=$ *sum required.*

10. To find the sum (S) of the infinite series of the reciprocals of the triangular numbers, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c.

Let $\dfrac{1}{1}+\dfrac{1}{3}+\dfrac{1}{6}+\dfrac{1}{10}$, &c. *ad infinitum* $=S$.

Or, $\dfrac{1}{1.1}+\dfrac{1}{1.3}+\dfrac{1}{2.3}+\dfrac{1}{2.5}$, &c. . . . . $=S$.

Then $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\dfrac{1}{4.5}$ &c. . . . . . . $=\dfrac{S}{2}$.

That is, $\dfrac{1}{1}-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{4}-\dfrac{1}{5}$, &c. $=\dfrac{S}{2}$.

$$\text{Or,}\left\{\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\ \&c.\\ \quad\ -\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}\ \&c.\end{array}\right\}=\frac{S}{2}.$$

$$\textit{Whence, } \frac{S}{2}=\frac{1}{1}\text{; or } S=2=\textit{sum required.}$$

* 11. To find the sum of n terms of the series $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c.

$$\textit{Let } z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},\ \&c.\ \textit{to}\ \frac{1}{n}.$$

$$\textit{Then } z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\ \&c.\ \textit{to}\ \frac{1}{n}$$

$$\textit{And } z-\frac{1}{1}+\frac{1}{n+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\ \&c.\ \textit{to}\ \frac{1}{n+1}$$

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* Let $\Sigma =$ sum of an infinite number of terms, and $S =$ sum of n terms.

Then the *formulæ* for the sums of the reciprocals of figurate numbers may be exhibited as follows:

		Σ	S
1	$\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}$, &c. $=$	∞	$= n.$
2	$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c. $=$	∞	$=$
3	$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c. $=$	$\frac{2}{1}$	$=\frac{2}{1}-\frac{2}{1}\times\frac{1}{n+1}$
4	$\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$, &c. $=$	$\frac{3}{2}$	$=\frac{3}{2}-\frac{3}{2}\times\frac{1.2}{n-1.n-2}$
5	$\frac{1}{1}+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}$, &c. $=$	$\frac{4}{3}$	$=\frac{4}{3}-\frac{4}{3}\times\frac{1.2.3}{n+1.n+2.n+3}$
	&c.	&c.	&c.

Therefore $\frac{1}{1}-\frac{1}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ *&c. to* $\frac{1}{n}-\frac{1}{n+1}$

Or, $\frac{n}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ *&c. to* $\frac{1}{n.\overline{n+1}}$.

Whence, $\frac{2n}{n+1}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$ *&c. to* $\frac{2}{n.\overline{n+1}}$.

Or, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$ *&c. to* $\frac{2}{n.\overline{n+1}}$ *(or n terms)*

$=\frac{2n}{n+1}=$ *sum required.*

12. Required the sum of the infinite series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, &c.

Let $z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, *&c. ad infinitum.*

Then $z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, *&c. by transposition.*

And $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, *&c. by subtraction.*

Or $1-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}$, *&c. by transposition.*

And $\frac{1}{2}=\frac{4}{1.4.3}+\frac{6}{2.9.4}+\frac{8}{3.16.5}$, *&c. by subtraction.*

Or $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}$, *&c.*

Whence $\frac{1}{2}\div2=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, *&c.*

Or, $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, *&c.* $=\frac{1}{2}\div2=\frac{1}{4}=$ *sum required.*

13. To find the ſum of n terms of the ſeries $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6}$, &c.

$$\textit{Let } z = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \text{, \&c. to } \frac{1}{n.\overline{n+1}}.$$

$$\textit{Then } z - \frac{1}{2} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}\text{, \&c. to } \frac{1}{n.\overline{n+1}}.$$

$$\textit{And } z - \frac{1}{2} + \frac{1}{\overline{n+1}.\overline{n+2}} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}\text{, \&c. to } \frac{1}{\overline{n+1}.\overline{n+2}}.$$

Therefore $\frac{1}{2} - \frac{1}{\overline{n+1}.\overline{n+2}} = \frac{2}{1.2.3} + \frac{2}{2.3.4} + \frac{2}{3.4.5}$, *&c.* (*continued to n terms*) *by ſubtraction.*

Whence $\frac{1}{4} - \frac{1}{2.\overline{n+1}.\overline{n+2}} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$, *&c.* (*continued to n terms*) *by diviſion.*

And conſequently, $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$, *&c. to n terms* $= \frac{1}{4} - \frac{1}{2.\overline{n+1}.\overline{n+2}}$ = *ſum required.*

14. Required the ſum (S) of the ſeries $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}$, &c. continued *ad infinitum.*

$$\textit{Let } x = \frac{1}{2} \textit{ and } S = \frac{z}{1+x}.$$

$$\textit{Then } \frac{z}{1+x} = x - x^2 + x^3 - x^4 + x^5\text{, \&c.}$$

$$\textit{And } z = \overline{1+x} \times \overline{x - x^2 + x^3 - x^4\text{, \&c.}}$$

$$=\begin{cases} x-x^2+x^3-x^4+x^5, \text{ \&c.} \\ 1+x \end{cases}$$

$$\begin{array}{l} x-x^2+x^3-x^4+x^5, \text{ \&c.} \\ \quad +x^2-x^3+x^4-x^5, \text{ \&c.} \\ \hline =x+0+0+0+0, \text{ \&c.} \end{array}$$

Therefore $z=x$.

And $x-x^2+x^3-x^4+x^5$, &c. $=\frac{x}{1+x}$.

Or $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}$, &c. $=\frac{\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{3}=$ *sum required.*

15. Required the sum of the infinite series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}$, &c.

Let $x=\frac{1}{2}$, *and* $S=\frac{z}{\overline{1-x}|^2}$.

Then $\frac{z}{\overline{1-x}|^2}=x+2x^2+3x^3+4x^4+5x^5$, &c.

And $z=\overline{1-x}|^2\times\overline{x+2x^2+3x^3+4x^4, \text{ \&c.}}$

$$=\begin{cases} x+2x^2+3x^3+4x^4, \text{ \&c.} \\ 1-2x+x^2 \end{cases}$$

$$\begin{array}{l} x+2x^2+3x^3+4x^4, \text{ \&c.} \\ \quad -2x^2-4x^3-6x^3, \text{ \&c.} \\ \qquad\quad + x^3+2x^4, \text{ \&c.} \\ \hline x+\ 0+\ 0+\ 0, \text{ \&c.} \end{array}$$

Therefore $x=z$.

And $x+2x^2+3x^3+4x^4+5x^5$, &c. $=\frac{x}{\overline{1-x}|^2}$

Or $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}$, &c. $=\frac{\frac{1}{2}}{\overline{1-\frac{1}{2}}|^2}=2=$ *ſum required.*

16. Required the ſum (S) of the infinite ſeries $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$, &c.

$$\textit{Let } x=\frac{1}{3}, \textit{ and } \frac{z}{\overline{1-x}|^3}=S.$$

$$\textit{Then } \frac{z}{\overline{1-x}|^3}=x+4x^2+9x^3+16x^4+25x^5, \textit{ \&c.}$$

$$\textit{And } z=\overline{1-x}|^3\times\overline{x+4x^2+9x^3+16x^4, \textit{ \&c.}}$$

$=x+x^2$, *as will be found by actual multiplication.*

$$\textit{Therefore } x+x^2=z.$$

And conſequently $x+4x^2+9x^3+16x^4$, *&c.* $=\frac{x\times\overline{1+x}}{\overline{1-x}|^3}$

Or $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$, *&c.* $=\frac{\frac{1}{3}\times\overline{1+\frac{1}{3}}}{\overline{1-\frac{1}{3}}|^3}=\frac{3}{2}=$ *ſum required.*

17. Required the ſum (S) of the infinite ſeries $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}$, &c.

$$\textit{Let } x=\frac{1}{r}, \textit{ and } S=\frac{z}{m.\overline{1-x}|^2}.$$

$$\textit{Then } \frac{z}{mr\,\overline{1-x}|^2}=\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}, \textit{ \&c.}$$

$$\textit{Or } \frac{z}{\overline{1-x}|^2}=a+\frac{a+d}{r}+\frac{a+2d}{r^2}+\frac{a+3d}{r^3}, \textit{ \&c.}$$

That is $\frac{z}{\overline{1-x}|^2}=a+\overline{a+d}\times x+\overline{a+2d}\times x^2+\overline{a+3d}\times x^3$, *&c.*

And $z = \overline{1-x}|^2 \times \overline{a + a+dx + a+2dx^2 + a+3dx^3}$, *&c.* $= \overline{1-x} \times a+dx$, *as will appear by actual multiplication.*

Therefore $z = \overline{1-x} \times a+dx$.

And consequently $\frac{a}{m} + \frac{a+d}{mr} + \frac{a+2d}{mr^2}$, *&c.* $=$

$\frac{\overline{1-x} \times a+dx}{m.\overline{1-x}|^2}$ = *sum required.*

EXAMPLES FOR PRACTICE.

1. To find the sum of n terms of the series $a + \overline{a-d} + \overline{a-2d} + \overline{a-3d} + \overline{a-4d}$, &c.

Ans. $\frac{n}{2} \times \overline{2a - \overline{n-1} \times d}$.

2. Required the sum of the infinite series $a + da + d^2a + d^3a + d^4a$, &c. where d is a proper fraction.

Ans. $\frac{a}{1-d}$.

3. To find the sum of the infinite series $1 + 2^4x + 3^4x^2 + 4^4x^3 + 5^4x^4$, &c. *Ans.* $\frac{1 + 11x + 11x^2 + x^3}{\overline{1-x}|^6}$.

4. Required the sum of the infinite series $1 + 3x + 6x^2 + 10x^3 + 15x^4$, &c.

Ans. $\frac{1}{\overline{1-x}|^2}$.

5. Required the sum of the infinite series $1 + 4x + 10x^2 + 20x^3 + 35x^4$, &c.

Ans. $\frac{1}{\overline{1-x}|^3}$.

6. Required the sum of the infinite series $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9}$, &c.

Ans. $\frac{1}{2}$.

7. Required the sum of 40 terms of the series $\overline{1\times2}+\overline{3\times4}+\overline{5\times6}+\overline{6\times7}$, &c. *Ans.* 22960.

8. Required the sum of the infinite series $\frac{1}{1.2.3.4.}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c. *Ans.* $\frac{1}{18}$.

9. Required the sum of n terms of the series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c.

Ans. $\frac{1}{18}-\frac{1}{3.\overline{n+1}.\overline{n+2}.\overline{n+3}}$.

10. Required the sum of n terms of the series $\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}$, &c.

Ans. $\frac{\overline{1-p}\times r}{\overline{r-1}|^2}-\frac{np}{r-1}$, *where* $p=\frac{1}{r}$.

11. Required the sum of the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}$, &c. $\frac{1}{n\times\overline{3+n}}$.

Ans. $\Sigma=\frac{11}{18}$, $S=\frac{n}{3+3n}+\frac{n}{12+6n}+\frac{n}{27+9n}$.

12. Required the sum of the series $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}$, &c. $\frac{1}{2n.\overline{4+2n}}$.

Ans. $\Sigma=\frac{3}{16}$, $S=\frac{5n+3n^2}{32+48n+16n^2}$.

13. Required the sum of the series $\frac{1}{4.8} - \frac{1}{6.10} + \frac{1}{8.12} -$, &c. $\frac{1}{\overline{2+2n}.\overline{6+2n}}$.

$$Ans.\ \Sigma = \frac{1}{48},\ S = \frac{n}{16+16n} - \frac{n}{36+24n}.$$

14. Required the sum of the series $\frac{1}{3.8} + \frac{1}{6.12} + \frac{1}{9.16} + \frac{1}{12.20}$, &c. $\frac{1}{3n.\overline{4+4n}}$.

$$Ans.\ \Sigma = \frac{1}{12},\ S = \frac{n}{12+12n}.$$

* 15. Required the sum of the series $\frac{1}{2.6.4.5} + \frac{1}{4.8.5.6} + \frac{1}{6.10.6.7}$, &c. $\frac{1}{2n.\overline{4+2n}.\overline{3+n}.\overline{4+n}}$.

$$Ans.\ \Sigma = \frac{7}{1152},\ S = \frac{n}{24} \times : \frac{1}{4.\overline{1+n}} + \frac{1}{8.\overline{2+n}} - \frac{5}{12.\overline{3+n}} + \frac{3}{16.\overline{4+n}}.$$

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* A great variety of series, of different forms, may be found in other authors; but those which are here given will be sufficient for the learner's practice.

The names of the principal authors, who have written upon this subject, are as follow:

Archimedes; Arabes; D'Alembert; Barrow; Briggs; Nicholas, Daniel, John and James Bernoulli; Fermat; De Cartes; Clairaut; Condercet; Cotes; Dodson; Euler; Emerson; Fagnanus; Le Grange; Goldbach; Gregory; Halley; Harriot; Huddens; Huygens; Hutton; Kepler; Keil; Landen; Mac Laurin; De Lagney; Leibnitz; Lorgna; Lucas de Burgo; Manfredi; Monmort; De Moivre; Montono; Nichole; Newton; Oughtred; Riccati; Regnald; Saunderson; Sterling; Slusius; Simpson; Brook Taylor; Varignon; Vieta; Wallis; Waring; &c.

OF LOGARITHMS*.

Logarithms are numbers ſo contrived and adapted to other numbers, that the ſums and differences of the former ſhall correſpond to, and ſhew, the products and quotients of the latter.

Or, more generally, logarithms are the numerical exponents of ratios; or a ſeries of numbers in arithmetical progreſſion, anſwering to another ſeries of numbers in geometrical progreſſion.

Thus	0.	1.	2.	3.	4.	5.	*Indices, or logarithms.*
	1.	2.	4.	8.	16.	32.	*Geometric progreſſion.*
Or	0.	1.	2.	3.	4.	5.	*Indices, or logarithms.*
	1.	3.	9.	27.	81.	243.	*Geometric progreſſion.*
Or	0.	1.	2.	3.	4.	5.	*Ind. or log.*
	1.	10.	100.	1000.	10000.	100000.	*Geo. prog.*

&c.

And, from hence, it is evident, that there may be as many kinds of indices, or logarithms, as there can be taken different kinds of geometric ſeries; and that in any ſyſtem, or table, of logarithms whatever, the logarithm of unity, or 1, will be nothing.

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* The invention of logarithms is the undoubted right of Lord *Neper*, Baron of *Merchiſton*, in *Scotland*, and is properly conſidered as one of the moſt uſeful and excellent diſcoveries of modern times. A table of theſe numbers was firſt publiſhed by him at Edinburgh, anno 1614, in a treatiſe entitled *Canon Mirificum Logarithmorum*; and as their great utility and extenſive application, were ſufficiently apparent, they were immediately received by all the learned throughout Europe. Mr. *Henry Briggs*, *Savilian* Profeſſor of Geometry at *Oxford*,

It is, alſo, apparent, from the nature of theſe ſeries, that, if any two indices be added together, their ſum will be the index of that number which is equal to the product of the two terms, in the geometic progreſſion, to which thoſe indices belong.

Thus, the indices 2 and 3, being added together, are =5; and the numbers 4 and 8, or the terms correſponding with thoſe indices, being multiplied together, are =32, which is the number anſwering to the index 5.

And, in like manner, if any one index be ſubtracted from another, the difference will be the index of that number which is equal to the quotient of the two terms to which thoſe indices belong.

Thus the index 6, minus the index 4, is =2; and the terms correſponding to thoſe indices are 64 and 16, whoſe quotient is =4; which is the number anſwering to the index 2.

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upon hearing of the diſcovery, ſet out on a viſit to the noble inventor, and ſoon afterwards they jointly undertook the arduous taſk of computing new tables upon this ſubject, and reducing them to a more convenient form than that which was at firſt thought of. But Lord *Neper* dying before they were finiſhed, the whole burden was laid upon Mr. *Briggs*, who with prodigious labour, and great ſkill, made an entire *Canon*, according to the new form, for all numbers from 1 to 20000, and from 9000 to 101000, to 14 places of figures, and publiſhed it at *London* in the year 1624, in a treatiſe entitled *Arithmetica Logarithmica*, with directions for ſupplying the intermediate *chiliads*.

This *Canon* was again publiſhed in *Holland*, by *Adrian Vlacq*, *anno* 1628, together with the logarithms of all the numbers which Mr. *Briggs* had omitted; but he continued them only to 10 places of decimals. Mr. *Briggs* alſo computed the logarithms of the ſigns, tangents and ſecants, to every degree, and $\frac{1}{100}$ part of a degree of the whole quadrant; and ſubjoined

For the ſame reaſon, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power.

Thus, the index, or logarithm of 4, *in the above ſeries, is* 2; *and if this number be multiplied by* 3, *the product will be* =6; *which is the logarithm of* 64, *or the third power of* 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root.

Thus the index, or logarithm of 64 *is* 6; *and if this number be divided by* 2, *the quotient will be* =3; *which is the logarithm of* 8, *or the ſquare root of* 64.

The logarithms moſt convenient for practice are ſuch as are adopted to a geometric ſeries increaſing

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them to the natural ſigns, tangents and ſecants, which he had before computed to 15 places of figures. And theſe tables, together with their conſtruction and uſe, were firſt publiſhed in the year 1633, after Mr. *Briggs*'s death, by Mr. *Henry Gellibrand*, under the title of *Trigonometria Britannica.*

Benjamin Urſinus has alſo given us a table of logarithms to every 10 ſeconds. And Mr. *Wolf*, in his *Mathematical Lexicon*, ſays, that one *Van Loſer* had computed them to every ſingle ſecond; but his untimely death prevented their publication.

A great number of other authors have treated of this ſubject, but as their numbers are frequently inaccurate, and incommodiouſly diſpoſed, they are now generally neglected. The tables in moſt repute at preſent, are thoſe of *Gardiner* in 4to, firſt printed in the year 1742, and *Sherwin* in 8vo, firſt printed in the year 1705, where the logarithms of all numbers may be eaſily found from 1 to 10000000; and thoſe of the ſigns, tangents, and ſecants, to any degree of accuracy required.

Dodſon's *Antilogarithmic Canon* is likewiſe a very ingenious work, and is of great uſe for finding the numbers anſwering to any given logarithms.

in a tenfold proportion, as in the laſt of the above examples; and are thoſe which are to be found, at preſent, in moſt of the common tables upon this ſubject.

And the diſtinguiſhing mark of this ſyſtem of logarithms is, that the index, or logarithm, of 1 is 0; that of 10, 1; that of 100, 2; that of 1000, 3, &c. And, from hence it follows, that the logarithm of any number between 1 and 10 muſt be 0 and ſome fractional parts; and that of a number between 10 and 100 will be 1 and ſome fractional parts; and ſo on for any other number whatever.

And ſince the integral part of a logarithm is always thus readily found, it is uſually called the *index*, or *characteriſtic*; and is commonly omitted in the tables; being left to be ſupplied by the operator himſelf, as occaſion requires.

Of the making of LOGARITHMS.

Whatever arithmetical progreſſion we apply to a geometrical one, the terms of it are logarithms only to that ſeries to which we apply them, and anſwer the end propoſed only for thoſe particular numbers; ſo that if we had logarithms adapted only to particular geometrical ſeries, they would be but of little uſe. The great end and deſign of logarithms is the eaſe and expedition they afford in long calculations, by ſaving the laborious work of *multiplication*, *diviſion*, and the *extraction of roots*; but this end would never be completely anſwered, unleſs logarithms could be adapted to the whole ſyſtem of numbers, 1, 2, 3, 4, &c. And as here lay the excellence and merit of the contrivance, ſo alſo the difficulty. For the natural

ſyſtem of numbers, 1, 2, 3, 4, &c. being an arithmetical, and not a geometrical ſeries, ſeems rather fit to be made logarithms of, than to have logarithms applied to it. Yet this difficulty may be eaſily removed, by conſidering,

That though the whole ſyſtem of natural numbers, 1, 2, 3, 4, &c. makes not one geometrical ſeries, and cannot, by any means, be brought within ſuch a ſeries of determinate numbers, yet they may be brought ſo near to it, as to be within any aſſignable degree of approximation; which may be conceived, in general, thus: Suppoſe a fraction indefinitely ſmall to be repreſented by x, and a geometrical ſeries ariſing from 1, in the ratio of 1 to $1+x$, to be 1, $\overline{1+x}|^1$, $\overline{1+x}|^2$, $\overline{1+x}|^3$, $\overline{1+x}|^4$, &c. Then muſt ſome of theſe terms come indefinitely near to coincide with all the natural numbers, 1, 2, 3, 4, &c.; becauſe amongſt numbers that ariſe by indefinitely ſmall increments, ſome of them muſt exceed, or fall ſhort, of any determinate number, by an indefinitely little exceſs or defect.

If, therefore, in the places of the terms of this ſeries, that do approach indefinitely near to any of the natural numbers, we ſuppoſe theſe natural numbers themſelves to be ſubſtituted, then will the ſeries be a geometrical progreſſion, to an exactneſs that may be called *indefinite*; becauſe the approximation of its terms to the natural numbers, can never end, but goes on *in infinitum.*

And ſince this imagined geometric ſeries comprehends, indefinitely near, the whole ſyſtem of natural numbers, 1, 2, 3, 4, &c. ſo the indices of its terms comprehend a whole ſyſtem of logarithms, which are adapted to this ſyſtem of numbers, and may be extended to any length we pleaſe. For though the natural ſyſtem of numbers make not, by themſelves,

a complete geometrical feries, yet they are conceived as a part of fuch a feries, and their logarithms are the indices of their diftances from unity in that feries; or, more generally, they are the correfponding terms of an arithmetical feries applied to that geometrical one.

But, again, it muft be obferved, that an indefinitely fmall fraction cannot be affigned; and, therefore, in the actual conftruction of logarithms, we muft be contented with a determinate degree of approximation. Whence, accordingly as we take x, fo in the feries $1, \overline{1+x}|^1, \overline{1+x}|^2, \overline{1+x}|^3, \overline{1+x}|^4$, &c. the approximation of its terms to the natural numbers will be in different degrees of exactnefs: for the lefs x is, the nearer will be the approximation; but then the more are the number of involutions of $1+x$, neceffary to come within any determinate degree of nearnefs to the natural number affigned.

Thus then we may conceive the poffibility of making logarithms to all the natural numbers, 1, 2, 3, 4, &c. to any determinate degree of exactnefs; viz. by affigning a very fmall fraction for x, and actually raifing a feries, in the ratio of 1 to $1+x$, and taking for the natural numbers fuch terms of that feries as are the neareft to them, and their indices for the logarithms. But then, to conftruct logarithms in this manner, to fuch an extent of numbers, and degree of exactnefs, as would be neceffary to make them of any confiderable ufe, is next to impoffible, becaufe of the almoft infinite labour and time it would require. This, however, is an introduction for underftanding the method of the *noble inventor*, who undoubtedly firft took the hint of making logarithms from the confideration of the indices of a geometrical feries; and by means

of the principles and known properties of these progressions, he first formed his tables, and adapted them to the practical purposes intended.

PROBLEM I.

To find the logarithm of any of the natural numbers, 1, 2, 3, 4, *&c. according to the method of* NEPER.

RULE.

1. Take the geometrical series, 1, 10, 100, 1000, 10000, &c. and apply to it the arithmetical series 1, 2, 3, 4, 5, &c, as logarithms.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the series betwixt which the number proposed lies.

3. Between the mean, thus found, and the nearest extreme, find another geometrical mean, in the same manner; and so on, till you are arrived within the proposed limit of the number whose logarithm is sought.

4. Find as many arithmetical means, in the same order as you found the geometrical ones, and the last of these will be the logarithm answering to the number required.

EXAMPLE:

Let it be required to find the logarithm of 9.

Here the numbers between which 9 *lies are* 1 *and* 10.

First, then, the log. of 10 *is* 1, *and the log. of* 1 *is* 0; *therefore* $\frac{1+0}{2}=.5$ *is the arithmetical mean. And* $\sqrt{1\times10}=\sqrt{10}=3.1622777=$ *geometric mean; whence the logarithm of* 3.1622777 *is* .5.

Secondly, the log. of 10 *is* 1, *and the log. of* 3.1622777 *is* .5; *therefore* $\frac{1+.5}{2}=.75=$ *arithmetical mean. And* $\sqrt{10\times3.1622777}=5.6234132=$ *geometric mean; whence the log. of* 5.6234132 *is* .75.

Thirdly, the log. of 10 *is* 1, *and the log. of* 5.6234132 *is* .75; *therefore* $\frac{1+.75}{2}=.875=$ *arithmetical mean. And* $\sqrt{10.\times5.6234132}=7.4989421$ $=$ *geometric mean; whence the log. of* 7.4989421 *is* .875.

Fourthly, the log. of 10 *is* 1, *and the log. of* 7.4989421 *is* .875; *therefore* $\frac{1+.875}{2}=.9375=$ *arithmeticalmean. And* $\sqrt{10\times7.4989421}=8.6596431$ $=$ *geometric mean; whence the log. of* 8.6596431 *is* .9375.

Fifthly, the log. of 10 *is* 1, *and the log. of* 8.6596431 *is* .9375; *therefore* $\frac{1+.9375}{2}=.96875$ $=$ *arithmetical mean. And* $\sqrt{10\times8.6596431}=$ 9.3057204 $=$ *geometric mean; whence the log. of* 9.3057204 *is* .96875.

Sixthly, the log. of 8.6596431 *is* .9375, *and the log. of* 9.3057204 *is* .96875; *therefore* $\frac{.9375+.96875}{2}=$ .953125 $=$ *arithmetical mean. And* $\sqrt{8.6596431\times9.3057204}=8.9768713=$ *geometric mean; whence the log. of* 8.9768713 *is* .953125.

And, proceeding in this manner, after 25 *extractions, the logarithm of* 8.9999998 *will be found to be* .9542425; *which may be taken for the logarithm of* 9, *because it differs from it only by* $\frac{1}{5000000}$, *and is therefore sufficiently exact for all practical purposes.*

And in the same manner the logarithms of almost all the prime numbers were found; a work so incredibly laborious, that the unremitted industry of several years was scarcely sufficient for its accomplishment.

PROBLEM II.

To determine the hyperbolic logarithm (L) of any given number (N).

The hyperbolic logarithm of any number, is the index of that term of the logarithmic progression, agreeing with the proposed number, multiplied by the excess of the common ratio above unity.

Let, therefore, $\overline{1+x}|^n$ be that term of the logarithmic progression, $1, \overline{1+x}|^1, \overline{1+x}|^2, \overline{1+x}|^3, \overline{1+x}|^4$, &c. which is equal to the required number (N).

Then will $\overline{1+x}|^n = N$, and $1+x = N^{\frac{1}{n}}$; and if $1+y$ be put $= N$, and $m = \frac{1}{n}$, we shall have $1+x = N^{\frac{1}{n}} = \overline{1+y}|^m = 1 + my + m \times \frac{m-1}{2} y^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} y^3$, &c.

And, consequently, $x = my + m \times \frac{m-1}{2} y^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} y^3$, &c. where m being rejected in the factors $m-1$, $m-2$, $m-3$, &c. as being indefinitely small in comparison of 1, 2, 3, &c. the equation will become $x = my - \frac{my^2}{2} + \frac{my^3}{3} - \frac{my^4}{4}$, &c.

Hence $\frac{x}{m}$ $(nx = L) = x - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}$, &c. = hyperbolic logarithm of N, as was required.

PROBLEM III.

The hyperbolic logarithm (L) of a number being given, to find the number (N) itself, answering thereto.

Let $\overline{1+x}|^n$ be that term of the logarithmic progression, 1, $\overline{1+x}|^1$, $\overline{1+x}|^2$, $\overline{1+x}|^3$, $\overline{1+x}|^4$, &c. which is equal to the required number N.

Then, because $\overline{1+x}|^n$ is universally $= 1 + nx + n \times \frac{n-1}{2} x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3$, &c. we shall also have $1 + nx + n \times \frac{n-1}{2} x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^3$, &c. $= N$.

But because n, from the nature of logarithms, is here supposed indefinitely great, it is evident that the numbers connected to it by the sign —, may all be rejected, as far as any assigned number of terms, being indefinitely small in comparison of n.

Therefore, by throwing out 1, 2, 3, &c. from the factors $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. we shall have $1 + nx + \frac{n^2x^2}{2} + \frac{n^3x^3}{2.3} + \frac{n^4x^4}{2.3.4}$, &c. $= N$.

But nx $(= L)$ is the hyperbolic logarithm of $\overline{1+x}|^n$, or N, by what has been before specified; and therefore $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4}$, &c. $= N =$ number required.

PROBLEM IV.

To determine the hyperbolic logarithm (L) of any given number (N), by an universally converging series.

The series $y - \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4}$, &c. is the most easy and natural that can be obtained; but, in determining the logarithms of large numbers, it is but of little use, since, in all such cases, it diverges instead of converging.

Let, therefore, the number whose logarithm you would find, be denoted by $\frac{1}{1-y}$, and also let $\overline{1+x}|^n$ be the term of the logarithmic progression agreeing with the proposed number.

Then $\overline{1+x}|^n = \frac{1}{1-y}$; or $1+x = \frac{1}{\overline{1-y}|^{\frac{1}{n}}} = \overline{1-y}|^{-\frac{1}{n}} = \overline{1-y}|^m$ (by putting $m = -\frac{1}{n}$) $= 1 - my + m \times \frac{m-1}{2} y^2 - m \times \frac{m-1}{2} \times \frac{m-2}{3} y^3$, &c.

And, if m be rejected in the factors $m-1$, $m-2$, $m-3$, as before, our equation will become $1+x = 1 - my - \frac{my^2}{2} - \frac{my^3}{3} - \frac{my^4}{4}$ &c.

Whence $y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4}$ &c. $= -\frac{x}{m} = nx =$ hyperbolic logarithm of $\frac{1}{1-y}$; which series, it is manifest, will constantly converge, let the value of $\frac{1}{1-y}$ be ever so great; because y will always be less than unity.

But it is to be observed, that this series, except in its signs, has exactly the same form with that

Of the method of using a table of Logarithms.

Having explained the method of making a table of the logarithms of numbers greater than unity, the next thing to be done is, to shew how the logarithms of fractional quantities may be found. And, in order to this, it may be observed, that as we have hitherto supposed a geometric series to increase from an unit on the right hand, so we may now suppose it to decrease from an unit towards the left; and the indices, in this case, being made negative, will still exhibit the logarithms of the terms to which they belong.

Thus, Log. $-3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3$, *&c.*
Num. $\frac{1}{1000} \quad \frac{1}{100} \quad \frac{1}{10} \quad 1 \quad 10 \quad 100 \quad 1000$, *&c.*

Where $+1$ *is the logarithm of* 10, *and* -1 *the logarithm of* $\frac{1}{10}$; $+2$ *the logarithm of* 100, *and* -2 *the logarithm of* $\frac{1}{100}$, *&c.*

And from hence it appears, that all numbers, consisting of the same figures, whether they be integral, fractional, or mixed, will have the decimal parts of their logarithms the same.

Thus, the logarithm of 5874 *being* 3.7689339, *the logarithms of* $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, *&c. part of it will be be as follows:*

Num.	Logarithms.
5874	3.7689339
587.4	2.7689339
58.74	1.7689339
5.874	0.7689339
.5874	−1.7689339
.05874	−2.7689339
.005874	−3.7989339

From this it alſo appears, that the *index*, or *characteriſtic*, of any logarithm, is always one leſs than the number of figures which the natural number conſiſts of; and this index is conſtantly to be placed on the left hand of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a line drawn before it, like a negative quantity in algebra.

Thus, a number having 1, 2, 3, 4, 5, *&c. integer places.*

The index of its log. is 0, 1, 2, 3, 4, &c.

And a fraction having a digit in the place of primes, ſeconds, thirds, fourths, &c.

The index of its logarithm will be —1, —2, —3, —4, *&c.*

It may alſo be obſerved, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative; and all operations are performed by them, in the ſame manner as by negative and affirmative quantities in algebra.

In taking out of a table the logarithm of any number, not exceeding 10000, we have the decimal part by inſpection; and if to this the proper characteriſtic be affixed, it will give the complete logarithm required.

But if the number, whoſe logarithm is required, be above 10000, then find the logarithm of the two neareſt numbers to it, that can be found in the table, and ſay, as their difference : the difference of their logarithms : : the difference between the neareſt number and that whoſe logarithm is required : the dif-

ference of their logarithms, *nearly*; and this difference being added to, or subtracted from, the nearest logarithm, according as it is greater or less than the required one, will give the logarithm required, *nearly* *.

Thus, let it be required to find the logarithm of 367182.

The decimal part of 3671 *is, by the table* .5647844; *and of* 3672 *is* .5649027;

$$\therefore \begin{array}{l}\textit{The} \\ \textit{log. of}\end{array} \left\{ \begin{array}{lcl} 367100 & \textit{is} & 5.5647844 \\ 367200 & \textit{is} & 5.5649027 \end{array} \right\}$$

Their diff. 100 .0001183 *diff.*

$$\begin{array}{l}\textit{Nearest } N^o \\ \textit{Given } N^o\end{array} \left\{ \begin{array}{l} 367200 \\ 367182 \end{array} \right.$$

.18 *diff.*

Therefore 100 : .0001183 :: 18 : .0000212,

And 5.5649027 — .0000212 = 5.5648815 = *logarithm of* 367182 *nearly*.

If the number consists both of integers and fractions, or is entirely fractional, find the decimal part of the logarithm as if all its figures were integral; and this, being prefixed to the proper characteristic, will give the logarithm required.

* This method being founded on the supposition, that the logarithms of all numbers between 367100 and 367200, increase, or decrease, equally, according to their distance from 367100 or 367200, is not strictly true, but nearly so; and the greater any numbers are with respect to their difference, the nearer will these differences be proportional. And, therefore, though this will not give the exact logarithm, yet it will be a very near approximation, and is sufficiently exact for most practical purposes.

To find the logarithm of a proper fraction; subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought; which, being that of a decimal fraction, must always have a negative index.

And to find the logarithm of a mixed number, reduce the given number into an improper fraction; then subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought.

In finding the number answering to any given logarithm, the index, if *affirmative*, will always shew how many integral places the required number consists of; and, if *negative*, in what place of decimals the first, or significant figure, stands; so that if the logarithm can be found in the table, the number answering to it will always be had by inspection.

But, if the logarithm cannot be exactly found in the table, find the next greater, and the next less, and then say, As the difference of these two logarithms : the difference of the numbers answering to them : : the difference between the given logarithm and the nearest tabular logarithm : a fourth number; which added to, or subtracted from, the natural number answering to the nearest tabular logarithm, according as that logarithm is less or greater than the given one, will give the number required, *nearly*.

Thus, let it be required to find the natural number answering to the logarithm 5.5648815.

The next less and greater logarithms, in the table, are

5.5647844	*The numbers*	367100
5.5649027	*answering*	367200
Their diff. .0001183		100 *diff.*

And 5.5649027—5.5648815=.0000212,
Therefore .0001183 : 100 : : .0000212 : 18 *nearly*.
Whence 367200—18=367182= *number required* *.

MULTIPLICATION *by* LOGARITHMS.

RULE.

Add the logarithms of the factors together, and their ſum will be the logarithm of the product required.

Obſerving to add what is to be carried from the decimal part of the logarithm to the ſum of the affirmative indices:

And that the difference between the affirmative and negative indices, is to be taken for the index to the logarithm of the product.

EXAMPLES.

1. Let the number 256 be multiplied by 4.

The log. of 256 = 2.4082400
The log. of 4 = 0.6020600

The product =1024 ...3.0103000

* Directions, at large, for the uſing of logarithms, may be found in moſt of the common tables upon this ſubject.— *Sherwin*'s Mathematical tables, of the *Edition* 1741, or 1742, are reckoned the moſt correct and convenient, for practical purpoſes, of any now extant.

2. Let the number 8.5 be multiplied by 10.

The log. of 8.5 =	0.9294189
The log. of 10 =	1.0000000
The product = 85	1.9294189

3. Let the number 46.75 be multiplied by .3275.

The log. of 46.75 =	1.6697816
The log. of .3275 =	—1.5152113
The product = 15.31	1.1849929

4. Multiply 3.768, 2.053, and .007693 continually together.

The log. of	3.768 =	0.5761109
The log. of	2.053 =	0.3123889
The log. of	.007693 =	—3.8860957
The product =	.059511 ...	—2.7745955

5. Multiply .5, .4, and .12, continually together.

The log. of	.5 =	—1.6989700
The log. of	.4 =	—1.6020600
The log. of	.12 =	—1.0791812
The product =	.024 ...	—2.3802112

DIVISION *by* LOGARITHMS.

RULE.

From the logarithm of the dividend ſubtract the logarithm of the diviſor, and the number agreeing to the remainder will be the quotient required.

But obſerve to change the index of the diviſor from negative to affirmative, or from affirmative to negative, and then the difference of the affirmative indices muſt be taken for the index to the logarithm of the quotient.

And, alſo, when an unit is borrowed, in the left hand place of the decimal part of the logarithm, add it to the index of the diviſor; but if it be negative ſubtract it; and let the index ariſing from thence be changed and worked with as before.

EXAMPLES.

1. Let the number 56 be divided by the number 4.

The log. of 56 = 1.7481880
The log. of 4 = 0.6020600

The quotient =14 .... 1.1461280

2. Let the number 50.75 be divided by the number .25.

The log. of 50.75 = 1.7054360
The log. of .25 = −1.3979400

The quotient = 203 ...... 2.3074960

3. Let the number 24 be divided by the number 80.

The log. of .24 = −1.3802112
The log. of .80 = 1.9030900

The quotient .003 .... −3.4771212

4. Let the number .01265 be divided by the number .35.

The log. of .01265 = −2.1020905
The log. of .35 = −1.7403627

The quotient =.023 .... −2.3617278.

Involution *by* Logarithms*.

RULE.

1. Seek the logarithm of the given number in the table.

2. Multiply the logarithm, thus found, by the index of the proposed power.

3. Find the number corresponding to the product, and it will be the power required.

Note, In multiplying a logarithm with a negative index, by any affirmative number, the product will always be negative.

But what is to be carried from the decimal part of the logarithm will always be affirmative:

* The rule of proportion is performed by adding the logarithms of the two last terms, and subtracting the logarithm of the first.

And therefore their difference will be the index of the product; and is constantly to be made of the same kind with the greater.

EXAMPLES:

1. Required the second power of the number 3.874.

The log. of 3.874	=	0.5881596
The index	=	2
The power = 15.01		1.1763192

2. Required the third power of the number 2.768.

The log. of 2.768	=	0.4421661
The index	=	3
The power = 21.21		1.3264983

3. Required the third power of the number .7916.

The log. of .7916	=	−1.8985058
The index	=	3
The power = .4961		−1.6955174

4. Required the twelfth power of the number 1.539.

The log. of 1.539	=	0.1872386
The index	=	12
The power = 176.6		2.2468632

EVOLUTION *by* LOGARITHMS.

RULE.

1. Seek the logarithm of the given number, in the table.

2. Divide the logarithm, thus found, by the denominator of the index of the root propoſed.

3. Find the number correſponding to this quotient, and it will be the root required.

Note, When the index of the logarithm, to be divided, is negative, and does not exactly contain the diviſor; increaſe it by ſuch a number as will make it exactly diviſible, and carry the units borrowed, as ſo many tens, to the left hand place of the decimal, and then divide as in whole numbers.

EXAMPLES:

1. Required the ſquare root of the number 225.

The log. of 225 = 2.3521825
Therefore 2) 2.3521825

The root = 15 ... 1.1760912

2. Required the ſquare root of the number 1501.

The log. of 1501 = 3.1763807
Therefore 2) 3.1763807

The root = 38.74 .... 1.5881903

3. What is the cube root of the number .166375?

The log. of .166375 = —1.2210881
Therefore 3) —1.2210881

The root = .55 ... —1.7403627

4. What is the ſquare root of the number .08162?

The log. of .08162 = —2.9117966
Therefore 2) —2.9117966

The root = .2857 ... —1.4558983

5. What is the twelfth root of the number 176.6?

The log. of 176.6 = 2.2469907
Therefore 12) 2.2469907

The root = 1.539 .... .1872492

MISCELLANEOUS QUESTIONS.

1. A perſon being aſked what o'clock it was, anſwered, that it was between 8 and 9, and that the hour and minute hands were exactly together; what was the time?

Anſ. $8^h : 43' : 38''\frac{2}{11}$.

2. Divide the number 50 into two ſuch parts, that $\frac{3}{4}$ of one part, added to $\frac{5}{6}$ of the other, may make 40. *Anſ.* 20 *and* 30.

3. What two numbers are thoſe, whoſe difference is 12, and their ſquares equal to each other?

Anſ. +6 *and* —6.

4. There is a certain number, conſiſting of two places, which is equal to the difference of the ſquares

of its digits; and if 36 be added to it the digits will be inverted; quære the number? *Anſ.* 48.

5. Given $x^3+y^2=31$, and $y^3+x^2=17$; to find x and y. *Anſ.* $x=3$ *and* $y=2$.

6. Given $y^3-xy=666$, and $x^3+xy=406$; to find x and y. *Anſ.* $x=7$ *and* $y=9$.

7. Given the ſum of three numbers, in harmonical proportion, $=26$; and their continued product $=576$; to find the numbers. *Anſ.* 12, 8, *and* 6.

8. What two numbers are thoſe, whoſe difference, ſum, and product, are to each other as the numbers 2, 3, and 5 reſpectively? *Anſ.* 2 *and* 10.

9. To find that number whoſe cube being ſubtracted from its ſquare ſhall leave the greateſt remainder poſſible? *Anſ.* $\frac{2}{3}$.

10. Required to find the leaſt 3 whole numbers, ſo that $\frac{3}{8}$ of the firſt, $\frac{5}{14}$ of the ſecond, and $\frac{7}{20}$ of the third, ſhall be all equal to each other.

Anſ. 280, 294, *and* 300.

11. Given $zx^3+xz^3=290$, and $x^4+z^4=641$; to find x and z. *Anſ.* $x=5$, *and* $z=2$.

12. Given the ſum of three numbers in continued geometrical progreſſion $=39$, and the ſum of their ſquares $=819$; to find the numbers.

Anſ. 3, 9, 27.

13. Required the feweſt number of weights, and the weight of each, that ſhall weigh from one pound to 29 hundred weight.

Anſ. 1, 3, 9, 27, 81, 243, 729, *and* 2155.

14. Required two numbers ſuch, that their ſum ſhall be equal both to their product and the difference of their ſquares.

Anſ. 2.618034 *and* 1.618034.

15. It is required to find 4 affirmative integers ſuch, that the ſquare of the greateſt may be equal to the ſum of the ſquares of the other three.

Anſ. 3, 4, 12, *and* 13.

16. If money be lent, at three per cent.
To thoſe who chuſe to borrow,
In what time ſhall I be worth a pound,
If I lend a crown to-morrow?

Anſ. 46.90036 *years, allowing comp. int.*

17. Required the two leaſt nonquadrate numbers, x and y, ſuch, that x^2+y^2, and x^3+y^3 ſhall be both ſquare numbers. *Anſ.* $x=364$, *and* $y=273$.

18. There are three numbers in geometrical proportion ſuch, that if the mean be ſubtracted from the ſum of the two extremes, the remainder multiplied by the ſum of the ſaid two extremes will be 91; but if that remainder be multiplied by the ſum of all the three numbers, the product will be 133; it is required to find the three numbers by a ſimple equation. *Anſ.* 4, 6, *and* 9.

19. To determine two numbers whoſe ſum ſhall be a cube, but their product and quotients ſquares.

Anſ. 4 *and* 4, 100 *and* 25, 900 *and* 100.

20. Required that arithmetical progreſſion whoſe number of terms is 10, ſum of the terms 185, and the ſum of the cubes of the terms 104525.

Anſ. 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

21. To divide a given number (N) into 4 ſuch parts, that if any other number (n) be added to the firſt part, deducted from the ſecond, multiplied by the third, and the fourth part divided thereby, the ſum, difference, product, and quotient, ſhall be all equal to each other.

Anſ. $\frac{Nn}{\overline{n+1}|^2}-n$, $\frac{Nn}{\overline{n+1}|^2}+n$, $\frac{N}{\overline{n+1}|^2}$ *and* $\frac{Nn\times n}{\overline{n+1}|^2}=$ [*parts required.*

22. Given $x^3y+y^3x=512500$, and $x^2y-y^2x=2500$; to find x and y. *Anſ.* $x=25$ *and* $y=20$.

23. Given $x+y+z=6$, $xy+xz+yz=11$, and $xyz=6$; to find x, y, and z.

Anſ. $x=3$, $y=1$, *and* $z=2$.

24. To find two numbers in the ratio of 5 to 7, and which being respectively divided by 9 and 13, shall leave 3 and 8 for remainders.

Ans. 210 *and* 294.

25. To find three numbers such, that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be $=62$; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third $=47$; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third $=38$. *Ans.* 24, 60, *and* 120.

26. Given $x+\frac{y}{2}=357$, $y+\frac{z}{3}=476$, $z+\frac{w}{4}=595$, and $w+\frac{x}{5}=714$; to find x, y, z, and w.

Ans. $x=190$, $y=334$, $z=426$, and $w=676$.

27. To find four numbers x, y, z, and w, having the product of every three given; viz. $xyz=231$, $xyw=420$, $yzw=1540$, and $xzw=660$.

Ans. $x=3$, $y=7$, $z=11$, *and* $w=20$.

28. To find four numbers in geometric proportion, whose sum is 15, and the sum of their squares 85. *Ans.* 1, 2, 4, 8.

29. To find three numbers, x, y, and z, when the product of each by the sum of the other two are given; viz. $x\times\overline{y+z}=48$, $y\times\overline{x+z}=39$, and $z\times\overline{x+y}=63$.

Ans. $x=4$, $y=3$, *and* $z=9$.

30. What number is that, which, being any how divided, the square of one part, when added to the other part, shall always be a square number?

Ans. 1 *only*.

31. Given $y^3+z=127$, $y^3+x=135$, and $x^3+y^3+z^3=1133$; to find x, y, and z.

Ans. $x=10$, $y=5$, *and* $z=2$.

32. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2+xz=580$; to find x, y, and z.

Ans. $x=9$, $y=3$, *and* $z=20$.

33. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$; to find x, y, z.

Anf. $x=10$, $y=17$, *and* $z=22$.

34. To find the leaft number, which being divided by 6, 5, 4, 3, and 2, fhall leave the remainders 5, 4, 3, 2, and 1 refpectively. *Anf.* 59.

35. To find three numbers fuch, that the fum or difference of any two of them fhall be fquare numbers. *Anf.* 1873432, 2399057, *and* 2288168.

36. To find two fquare numbers fuch, that their fum may be a fquare, and their difference a cube, and the fide of the faid fquare and cube equal to each other.

Anf. $\frac{783}{15625}$ *and* $\frac{414}{15625}$.

37. To determine the number of fifteens that can be made out of a common pack of 52 cards.

Anf. 17264.

38. To find a fraction fuch, that being taken from its reciprocal the remainder fhall be a fquare.

Anf. Find fuch a fraction as that its biquadrate being added to 4 is a fquare, and it will anfwer the queftion.

39. Given $x^2+xy+y^2=1087$, and $x^4+x^3y^3+y^4=4577295$; to find x and y.

Anf. $x=21$ *and* $y=17$.

40. Given $x+y+z=78$, $x^2+y^2+z^2=2546$, and $xy-xz-yz=527$; to find x, y, and z.

Anf. $x=41$, $y=28$, *and* $z=9$.

41. Given $x+y=152$, and $\overline{x-y}|^{\frac{2}{3}} \times \overline{x-y}|^{\frac{3}{2}} = 8192$; to find x and y. *Anf.* $x=108$, *and* $y=44$.

42. To find three numbers fuch, that if to the fquare of each the product of the other 2 be added, the fums fhall be fquares. *Anf.* 73, 9, 328.

43 Let the number of cards in a pack (p) be diftributed into any number of heaps (n), by laying as many cards upon the bottom heap as are fuffi-

cient to make up its number q; then by having the number of cards remaining in the dealer's hand, (r) and the number of heaps (n) given, it is required to find the ſum of all the bottom cards.

Anſ. $\overline{q+1} \times n + \overline{r-p} =$ *ſum required.*

44. To find 3 numbers ſuch, that if each be ſubtracted from the cube of their ſum, the remainders ſhall be cubes. *Anſ.* $\frac{13851}{85184}$, $\frac{19467}{85184}$, *and* $\frac{18954}{85184}$.

45. Given $x^x = 123456789$ to find x.

Anſ. $x = 8.6400268$.

46. Given the cycle of the ſun 18, the golden number 8, and the Roman indiction 10; to find the year. *Anſ.* 1717.

47. To find 3 cube numbers ſuch, that their ſum ſhall be both a ſquare and a cube number; and if that ſum be ſquared it ſhall be a cube, and if it be cubed it ſhall be a ſquare.

Anſ. $\frac{x^6}{8}$, $\frac{8x^6}{27}$, $\frac{125x^6}{216}$; *where x may be any number whatever; if it be* $=216$ *they will be whole numbers.*

48. To find 3 numbers ſuch, that if each be added to the cube of their ſum, the ſums ſhall be cubes. *Anſ.* $\frac{23625}{157464}$, $\frac{1538}{157464}$, $\frac{18577}{157464}$.

49. With guineas and moidores, the feweſt, which way,
Three hundred and fifty-one pounds can you pay?
If paid every way 'twill admit of, what ſum
Do the pieces amount to?—my fortune's to come.

Anſ. 9 *guineas and* 253 *moidores; and* 37 *ways, which is* $=12987\,l.$

50. Given $x^{z^{\frac{1}{2}}} = z^{x^{\frac{2}{3}}} = 100$; to find x and z.

Anſ. $x = 47.706$ *and* $z = 1.42$.

51. Given $44000x^2 + 1 = z^2$; to find x and y in whole numbers.

Anſ. $x = 40482981221781$ *and* $z = 8491781781142001$.

52. To find three whole numbers ſuch, that the exceſs of the greateſt above the middle number, ſhall be to the exceſs of the middle number above the leaſt, as 3 to 1; and alſo that the ſum of every two of theſe ſhall be a ſquare.

Anſ. 1362, 402, 82; *or* $4^n \times 1362$, $4^n \times 402$, *and* $4^n \times 82$; *where* n *is any affirmative integer.*

53. Given $x + y = a\ (2)$, and $x^9 + y^9 = b\ (32)$, to find x and y by quadratics.

Anſ. $x = 1.4697175$ *and* $y = .5302824$.

54. Given $x^y = 500$, and $y^x = 300$; to find x and y. *Anſ.* $x = 4.6914$ *and* $y = 5.5102$.

55. Given $xy \times \overline{x+z}^{\,2} = 300$, $xz \times \overline{y+z}^{\,2} = 1296$, and $yz \times \overline{x+y}^{\,2} = 432$; to find x, y, and z.

Anſ. $x = 1$, $y = 3$, $z = 9$.

56. Given $w^3 + x + y + z = 57$, $w + x^3 + y + z = 2763$, $w + x + y^3 + z = 1353$, and $w + x + y + z^3 = 153$; to find x, y, z, and w.

Anſ. $x = 14$, $y = 11$, $z = 5$, *and* $w = 3$.

57. Given $x + y = 1750$, $xz + yv = 22708$, $xv + yz = 12292$, and $xzv + vzy = 159252$; to find x, y, z, and v. *Anſ.* $x = 1743$, $y = 7$, $z = 13$, *and* $v = 7$.

58. Given $5x + 7y + 9z = 93256$; to find all the different ſolutions in affirmative integers which the equation will admit of. *Anſ.* 13801148.

59. To find a ſquare number ſuch, that the ſum of all its aliquot parts ſhall be a ſquare number.

Anſ. 2401.

60. To find two ſquare numbers ſuch, that either of them, when added to its aliquot parts, ſhall make the ſame ſum. *Anſ.* 106276 *and* 165649.

61. To find three cube numbers ſuch, that their ſum may be both a ſquare and cube number.

Anſ. 1, $\frac{2048383}{274625}$, *and* $\frac{15252992}{274626}$.

62. To find 4 whole numbers ſuch, that the difference of every two ſhall be a ſquare number.

Anſ. 1873432, 2288168, 2399057, *and* 6560657.

F I N I S.

his definitions and rules as concise as possible, consistent with that simplicity and clearness which is absolutely necessary in things of this nature; and afterwards to exemplify those rules with a sufficient number of examples; in selecting of which, he has made choice of such as are most likely to occur in business; and has also shewn, with great clearness and perspicuity, the reason of each rule in notes; and, in some instances, has illustrated and explained the examples, when he had reason to apprehend any difficulties would be found; or where any disputes have arisen between former authors; and, in this part of his work, Mr. Bonnycastle has shewn great ingenuity and judgment.

By confining every thing of this nature to the notes, Mr. B. has been enabled to keep his text free from long explanations, so that nothing is to be found there, but what the Learner ought to transcribe, and fix in his memory; a matter which seems to have been too much neglected, by most of those authors who have undertaken to write on the subject of Arithmetic, for the use of Schools.

On the whole, we shall not hesitate to declare, that we think this little book will be found very useful both to the Teacher and Learner.

MONTHLY REVIEW.

THE Title of this little book we have given at full length, because it answers to its title, and does not, like many publications of this kind, profess more than it performs. The author has availed himself of *Malcolm's Arithmetic*, and has here given us a scientific, as well as practical treatise of this useful branch of Learning; so that students of every class may have their desires thoroughly gratified. The Book and its Author, of whom we know nothing, but from this performance, we recommend to the protection of the Public.

LONDON REVIEW.